

Geometry of Higgs bundles over elliptic curves related to automorphisms of simple Lie algebras, Calogero-Moser systems and KZB equations

A. Levin^{♭♯} M. Olshanetsky[§] A. Zotov^{◇♯♯}

§ – Institute for Information Transmission Problems, 127994 Moscow, Russia

♭ – NRU HSE, Department of Mathematics, Myasnitskaya str. 20, Moscow, 101000, Russia

♯ – ITEP, B. Cheremushkinskaya str. 25, Moscow, 117218, Russia

♯ – MIPT, Institutskii per. 9, Dolgoprudny, Moscow region, 141700, Russia

◇ – Steklov Mathematical Institute RAS, Gubkina str. 8, Moscow, 119991, Russia

E-mails: alevin@hse.ru, olshanet@itep.ru, zotov@mi.ras.ru

Abstract

We construct twisted Calogero-Moser (CM) systems with spins as the Hitchin systems derived from the Higgs bundles over elliptic curves, where transitions operators are defined by an arbitrary finite order automorphisms of the underlying Lie algebras. In this way we obtain the spin generalization of the twisted D'Hoker- Phong and Bordner-Corrigan-Sasaki-Takasaki systems. As by product, we construct the corresponding twisted classical dynamical r-matrices and Knizhnik-Zamolodchikov-Bernard equations related to the automorphisms of the Lie algebras.

Contents

1	Introduction	2
2	General construction	3
2.1	Holomorphic bundles over elliptic curves	3
3	The finite order automorphisms of simple Lie algebras	5
3.1	Inner and outer automorphisms	5
3.2	Generic finite order automorphisms automorphisms	5
3.3	Loop algebras and loop group	6

4	Holomorphic bundles over elliptic curves and automorphisms of Lie algebras	7
4.1	Adjoint bundles	7
4.2	Holomorphic bundle with non-diagonalized automorphisms	12
4.2.1	Special basis in $\mathfrak{sl}(N, \mathbb{C})$	13
4.2.2	Bundle $\mathcal{A}d_{SL}^{\nu, \Lambda}(C_{\bar{q}})$	14
4.3	Twisted vector bundles	15
5	Higgs bundles	18
5.1	Higgs bundles related to $\mathcal{A}d_G$ and $E_{\bar{G}}$ bundles	18
5.2	Hamiltonians	21
5.3	CM system corresponding to $\mathcal{A}d_{SL}^{\nu, \Lambda}(C_q)$ bundle	22
6	Classical r-matrices and KZB equations	23
6.1	r -matrix	23
6.2	KZB equations	24
7	Appendix A: Structure of Table 2	24
8	Appendix B: Some elliptic functions [2, 36]	26
9	Appendix C: Outer automorphisms of simple algebras [22, 23, 35]	29

1 Introduction

We consider a class of integrable systems similar to those that introduced some time ago in two cycles of works [4] and [19]. It is so-called twisted elliptic Calogero-Moser (CM) models related to a simple group G . Implicitly, they were appeared independently in the work of Etingof and Schiffmann [8] devoted to classification of dynamical classical r -matrices. This type of r -matrices were introduced in [10]. In this paper we identify these system with a specific class of the Hitchin system related to elliptic curves Σ_{τ} (τ is the modular parameter) and in this way associate them with special Higgs bundles over elliptic curves.

It was discovered in [17] (see also [7, 32]) that the standard SL_N CM systems can be considered as a Hitchin system related to the trivial Higgs bundle over an elliptic curve with a marked point. This approach was generalized to elliptic CM systems related to arbitrary simple Lie groups and topologically non-trivial Higgs bundles in [27, 28]. Simultaneously, the dynamical classical r -matrices were derived in the similar way via the Poisson reduction from the canonical structure on the Higgs bundles (see also the earlier works [5, 6]). The underlying holomorphic bundles in this construction are defined by two transition operators, satisfying some commutativity conditions. If their group commutator is the identity element, then the bundle is topologically trivial. In this case we come to the elliptic CM systems with spin [16, 21, 37]. If the group commutator is equal to a non-identity element of the center $\mathcal{Z}(G)$, the bundle is nontrivial. It is described by the characteristic classes defined as an element of $H^2(\Sigma_{\tau}, \mathcal{Z}(G))$. The moduli space of these bundles, the corresponding elliptic CM systems and dynamical r -matrices were considered in [27, 28].

In this paper we replace one of the transition operators by the finite-order automorphisms of the corresponding Lie algebras. We describe the moduli spaces of the trivial and nontrivial holomorphic bundles obtained in this way in terms of the quotient of Cartan algebras with respect to the action of the Bernshtein-Schwarzman group (4.32), (4.53), (4.67). We construct

the corresponding Higgs bundles and twisted CM systems.¹ In addition to the Lax operators and the Hamiltonians (5.23), (5.25) we derive the classical dynamical r -matrices (6.2). They govern the Poisson structures of the corresponding systems. In such a way we partly cover the classification of the classical dynamical r -matrices [8] based on symmetries of the extended Dynkin diagrams. The moduli spaces of the dynamical r -matrices are defined by the moduli spaces of the corresponding holomorphic bundles. They were introduced and studied in [9]. The r -matrices related to all finite automorphisms of simple Lie algebras coming from the trivial bundle were constructed in [10].

In particular, for the CM systems we come to results closed to [4] and [19]. It should be noted that the systems considered in those papers are spinless CM systems related to a simple group G , while in our case we come to the CM systems with spin. They can be considered a generalization of CM type systems described in [21]. To come to the spinless systems one should specify representations of the underlying Lie algebra, as it was done in [4, 19]. Another approach to the spinless systems developed in [20] is based on the equivariant embedding of G in $GL(N)$. Note, that the Hitchin type derivation of the twisted Calogero systems was proposed also in [24].

Following the general prescription [13, 15] we construct the Knizhnik-Zamolodchikov-Bernard (KZB) connection using the classical r -matrices. It is a flat connection in the bundle of conformal blocks over the moduli space of elliptic curves with marked points. The conformal blocks define the correlators in the WZW theory on elliptic curves. The fields in the theory considered in this paper satisfy the specific quasi-periodicity conditions. The monodromy of the fields around one of the cycles is defined by the outer automorphism of the underlying simple algebra. This construction is very similar to one proposed in [25, 29].

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2 General construction

Let Σ_g be a Riemann surface of genus g , G is a simple complex Lie group, and \mathfrak{g} its Lie algebra. Let \mathcal{P} be a principal G -bundle over Σ_g and $E_G = \mathcal{P} \times_G V$ is the associated bundle (V is a G module).

We will pay a special attention on the case, when V is the Lie algebra \mathfrak{g} with the adjoint action Ad_G of G on \mathfrak{g} . In this case we deal with the adjoint bundle $Ad_G = \mathcal{E}_G \times_{Ad_G} \mathfrak{g}$. The fibers of the bundle carry the Lie algebra \mathfrak{g} structure.

The Higgs bundle $\mathcal{H}(G, \Sigma_g)$ over the Riemann surface Σ_g is the pair (E_G, Φ) , where Φ is a Higgs field. It is a holomorphic 1-form on Σ_g taking values in $End E_G$.

Here we investigate the Higgs bundles over the elliptic curves (the genus one case) and construct the corresponding integrable systems.

2.1 Holomorphic bundles over elliptic curves

Our first goal is to describe the moduli spaces and to classify the topological types of these bundles, when Σ_g is an elliptic curve $\Sigma_\tau \sim \mathbb{C}/(\tau\mathbb{C} \oplus \mathbb{C})$, ($Im \tau > 0$).

The moduli spaces $Bun(\Sigma_\tau, G)$ can be defined in the following way [1, 3, 30]. Let ρ_1 and ρ_τ be two generators of the fundamental group $\pi_1(\Sigma_\tau)$ corresponding to the shifts $z \rightarrow z + 1$ and

¹For the trigonometric (Calogero-Sutherland) systems the similar approach was developed in [11, 12].

$z \rightarrow z + \tau$ satisfying the relation

$$\rho_1 \rho_\tau \rho_1^{-1} \rho_\tau^{-1} = 1. \quad (2.1)$$

The sections of a G -bundle $E_G(V)$ over Σ_τ satisfy the quasi-periodicity conditions

$$\psi(z+1) = A(z)\psi(z), \quad \psi(z+\tau) = B(z)\psi(z), \quad (\psi \in \Gamma(E_G(V))), \quad (2.2)$$

where the transition operators (A, B) should respect (2.1)

$$A(z)B^{-1}(z)A^{-1}(z+\tau)B(z+1) = 1. \quad (2.3)$$

By definition, these transition operators define a topologically trivial bundle.

Let ζ be an element of the center $\mathcal{Z}(\bar{G})$ of the universal covering group \bar{G} . To come to a non-trivial bundle replace (2.3) by equation

$$A(z)B^{-1}(z)A^{-1}(z+\tau)B(z+1) = \zeta. \quad (2.4)$$

A bundle \tilde{E} is equivalent to E if its sections $\tilde{\psi}$ are related to ψ as $\tilde{\psi}(z) = f(z)\psi(z)$, where $f(z)$ is invertible operator acting in V . It follows from (2.2) that transformed transition operators have the form:

$$A^f(z) = f(z+1)A(z)f^{-1}(z), \quad B^f(z) = f(z+\tau)B(z)f^{-1}(z). \quad (2.5)$$

These transformations form the group of automorphisms \mathcal{G} (the gauge group) of the bundle E_G .

According to [31] these the transition operators can be transformed to constants (see below Proposition 4.2). Thereby, (A, B) form the projective representation of $\pi_1(\Sigma_\tau)$. The moduli space of stable holomorphic bundles over Σ_τ can be defined as

$$\boxed{Bun_\zeta(\Sigma_\tau, G) = \{[A, B^{-1}] = \zeta\} / \mathcal{G}, \quad \zeta \in \mathcal{Z}(G).} \quad (2.6)$$

It was proved in [3, 30] that $Bun_\zeta(\Sigma_\tau, G)$ for $\zeta = 1$ is isomorphic to a weighted projective space.

Consider the universal covering group \bar{G} and the quotient group $G^{ad} = \bar{G} / \mathcal{Z}(\bar{G})$, where $\mathcal{Z}(\bar{G})$ is the center of \bar{G}

$$1 \rightarrow \mathcal{Z}(\bar{G}) \rightarrow \bar{G} \rightarrow G^{ad} \rightarrow 1. \quad (2.7)$$

Then for the analytic sheaves over Σ_τ we write the exact sequences

$$1 \rightarrow \mathcal{Z}(\bar{G}) \rightarrow \bar{G}(\mathcal{O}_\Sigma) \rightarrow G^{ad}(\mathcal{O}_\Sigma) \rightarrow 1,$$

Then we come to the long exact sequences

$$H^1(\Sigma_\tau, \mathcal{Z}(\bar{G})) \rightarrow H^1(\Sigma_\tau, \bar{G}(\mathcal{O}_\Sigma)) \rightarrow H^1(\Sigma_\tau, G^{ad}(\mathcal{O}_\Sigma)) \rightarrow H^2(\Sigma_\tau, \mathcal{Z}(\bar{G})) \sim \mathcal{Z}(\bar{G}) \rightarrow 0, \quad (2.8)$$

The first cohomology group $H^1(\Sigma_\tau, G(\mathcal{O}_\Sigma))$ of Σ_τ with coefficients in analytic sheaves defines the moduli space $Bun_\zeta(\Sigma_\tau, G)$ of the E_G -bundles. The elements from $H^2(\Sigma_\tau, G)$ are obstructions to lift the G^{ad} bundles to \bar{G} bundles. They provide the topological classification of the holomorphic vector bundles

$$\zeta(E_{G^{ad}}) \in H^2(\Sigma_\tau, \mathcal{Z}(\bar{G})) - \text{obstructions to lift } E_{G^{ad}} - \text{bundle to } E_{\bar{G}} - \text{bundle}.$$

Definition 2.1 Images of $H^1(\Sigma_\tau, G(\mathcal{O}_\Sigma))$ in $H^2(\Sigma_\tau, \mathcal{Z})$ are called the characteristic classes $\zeta(E_G)$ of G -bundles.

The cotangent bundle $T^*Bun(\Sigma_g, G)$ to the moduli space is the moduli space of the Higgs bundle over Σ_g . It is obtained from $\mathcal{H}(G, \Sigma_g)$ via the symplectic reduction. It is the phase space of algebraically integrable system [18]. In the case $T^*Bun_\zeta(\Sigma_\tau, G)$ we obtain the Calogero-Moser type systems [17, 27, 28]. To derive in this way the twisted Calogero-Moser systems and twisted Knizhnik-Zamolodchikov-Bernard equations we replace the transition operator A (2.2) on an outer automorphism of the underlying Lie algebra.

3 The finite order automorphisms of simple Lie algebras

3.1 Inner and outer automorphisms

The group of finite order automorphisms of simple complex Lie algebra \mathfrak{g} is a semidirect product of the subgroup of finite order inner automorphisms $Inn(\mathfrak{g})$ and the outer automorphisms $Out(\mathfrak{g})$

$$1 \rightarrow Inn(\mathfrak{g}) \rightarrow Aut(\mathfrak{g}) \rightarrow Out(\mathfrak{g}) \rightarrow 1.$$

Therefore, the outer automorphisms form the quotient group $Out(\mathfrak{g}) = Aut(\mathfrak{g})/Inn(\mathfrak{g})$.

Let G be a simple complex Lie group and $Lie(G) = \mathfrak{g}$. The outer automorphism group $Out(G)$ of the group G is the quotient $Aut(G)/Inn(G)$, where

$$Inn(G) = \{\iota_h g \rightarrow hgh^{-1}\} \sim G^{ad}$$

is the subgroup consisting of inner automorphisms. The outer automorphism group $Out(G)$ is dual to the center $\mathcal{Z}(G)$ in the following sense. The inner automorphism ι of G leads to the short exact sequence:

$$\mathcal{Z}(G) \hookrightarrow G \xrightarrow{\iota} Aut_G \rightarrow Out_G.$$

Here $\mathcal{Z}(G) = Ker(\iota)$ and $Out(G) \sim Cokernel(\iota)$.

The $Aut(\mathfrak{g})$ action has the following description [22, 23, 35]. Let us fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and system of simple roots $\Pi = \Pi(\mathfrak{g})$. Define group $Aut(\mathfrak{g}, \mathfrak{h}, \Pi)$ of automorphisms preserving \mathfrak{h} and Π . Let $H^{ad} = \exp ad(\mathfrak{h})$ be a maximal torus in the group G^{ad} and $Aut(\Pi)$ is a group of symmetries of the Dynkin diagram corresponding to Π . Then

$$Aut(\mathfrak{g}, \mathfrak{h}, \Pi) = H^{ad} \ltimes Aut(\Pi), \quad (3.1)$$

and the general automorphisms are represented in the form $Aut(\mathfrak{g}) = Aut(\mathfrak{g}, \mathfrak{h}, \Pi) \cdot Inn(\mathfrak{g})$.

Our goal is to construct a basis in \mathfrak{g} compatible with the $Aut(\mathfrak{g})$ action. It is presented in Appendix C.

3.2 Generic finite order automorphisms

In this subsection we follow [22, 23] (see also Appendix C).

We define the general automorphism of order p using (3.1). Let \mathfrak{g}_0 be the invariant subalgebra of ν (C.13) and \mathfrak{h}_0 its Cartan subalgebra. Using this structure we can represent the automorphism $\sigma \in Aut(\mathfrak{g}, \mathfrak{h}, \Pi)$ (3.1) in the form

$$\sigma = \nu \exp \left(\frac{2\pi i}{p} ad_{\kappa} \right), \quad \kappa \in \mathfrak{h}_0. \quad (3.2)$$

Let (s_0, s_1, \dots, s_n) be nonnegative integers without nontrivial common factor and

$$p = r \sum_{j=0}^n a_j s_j, \quad (\nu^r = Id), \quad (3.3)$$

(see (C.8)). The general automorphism $\sigma_{r,s}$ of \mathfrak{g} is defined by its action on the root subspaces $(E_{\tilde{\alpha}_k}, \tilde{\alpha}_k \in \tilde{\Pi}, (C.7), E_{-\tilde{\alpha}_0}, (C.8))$

$$\sigma_{r,s} E_{\tilde{\alpha}_k} = e(rs_k/p) E_{\tilde{\alpha}_k}, \quad k = 0, \dots, n. \quad (3.4)$$

If $\nu = Id$ it is the adjoint action of the invariant torus $H_0^{ad} = \exp\left(\frac{2\pi i}{p} ad_\kappa\right)$, ($\kappa \in \mathfrak{h}_0$) on \mathfrak{g} . Define the element $\kappa \in \mathfrak{h}_0$ by its expansion in the coroots basis H_j (C.27)

$$\kappa = \sum_{j=1}^n c_j H_j \in \mathfrak{h}_0,$$

such that $Ad_{e(\kappa)} = \sigma_{1,s}$. Then the coefficients c_j are defined as follows. Let $(\tilde{a})_{jk} = \langle \tilde{\alpha}_k, H_j \rangle$ be the Cartan matrix corresponding to the twisted root system $\tilde{\Pi}$ (C.7). Then $c_j = (\tilde{a}^{-1})_{jk} s_k r / m$ and

$$\mathcal{Q} = e(\kappa) = \exp\left(2\pi i \frac{r}{p} \sum_{j,k=1}^n (\tilde{a}^{-1})_{jk} s_k H_j\right). \quad (3.5)$$

Then $Ad_{\mathcal{Q}} E_{\tilde{\alpha}_k} = e\left(\frac{r}{p} s_k\right) E_{\tilde{\alpha}_k}$. It follows from (C.8) and (C.3) that $e\left(\frac{r}{p} \sum_{k=1}^n a_k s_k\right) = e\left(-\frac{r}{p} a_0 s_0\right)$. Thus,

$$\sigma_{1,s} E_{-\tilde{\alpha}_0} \equiv Ad_{\mathcal{Q}} E_{-\tilde{\alpha}_0} = e(s_0 r / p) E_{-\tilde{\alpha}_0} \quad (3.6)$$

and in this way $\sigma_{r,s}$ really defines the automorphism of order p . It is generated by the action (3.2)

$$\sigma_{r,s} = \nu \mathcal{Q} = \nu e(\kappa), \quad \nu \in Out(\mathfrak{g}), \quad \nu^r = Id. \quad (3.7)$$

The automorphism $\sigma_{r,s}$ defines $\mathbb{Z}/p\mathbb{Z}$ -gradation of the Lie algebra \mathfrak{g}

$$\mathfrak{g} = \sum_j \mathfrak{g}_j, \quad \sigma_{r,s}(\mathfrak{g}_j) = \omega \mathfrak{g}_j, \quad \omega^p = 1. \quad (3.8)$$

3.3 Loop algebras and loop group

Let $\mathcal{L}(\mathfrak{g}, t) = \mathfrak{g} \otimes C[t^{-1}, t]$ be the loop algebra. It is a map $\mathbb{C}^* \rightarrow \mathfrak{g}$. Define the action of the automorphism $\sigma_{r,s}$ on t as $\sigma_{r,s}(t) = \omega t$. Consider equivariant maps $\mathcal{L}_0(\mathfrak{g}, t, \sigma_{r,s}) \in \mathbb{C}^* \rightarrow \mathfrak{g}$ with respect to $\sigma_{r,s}$

$$\mathcal{L}_0(\mathfrak{g}, t, \sigma_{r,s}) = \mathcal{L}_0(\sigma_{r,s}(\mathfrak{g}), \omega^{-1}t), \quad \mathcal{L}_0(\mathfrak{g}, t, \nu) = \sum_j t^j \mathfrak{g}_{j, mod p}, \quad (3.9)$$

where \mathfrak{g}_j is defined in (3.8).

Let $L(G)$ be the loop group corresponding to the G

$$L(G) = G \otimes C[t^{-1}, t] = \left\{ \sum_k g_k t^k, \quad g_k \in G \right\}. \quad (3.10)$$

Let N^- be the negative unipotent subgroup of G and B is the positive Borel subgroup. Define the loop subgroups

$$L^+(G) = \{g_0 + g_1 t + \dots = g_0 + t L[t]\}, \quad g_j \in G, \quad g_0 \in B, \quad (3.11)$$

$$N^-(G) = \{n_- + g_1 t^{-1} + \dots = n_- + t^{-1} L[t^{-1}]\}, \quad n_- \in N^-. \quad (3.12)$$

Consider two types of the affine Weyl groups:

$$W_P = \{\hat{w} = wt^\gamma, \quad w \in W, \quad \gamma \in P^\vee\}, \quad W_Q = \{\hat{w} = wt^\gamma, \quad w \in W, \quad \gamma \in Q^\vee\}. \quad (3.13)$$

They act on the root vectors as $E_{\hat{\alpha}} = E_{\alpha} t^n \rightarrow E_{\hat{w}(\hat{\alpha})} = E_{w(\alpha)} t^{n+\langle \gamma, \alpha \rangle}$. For the loop groups we have the affine Bruhat decompositions [34]

$$L(G^{ad}) = \bigcup_{w \in \tilde{W}_P} N^-(G^{ad}) \hat{w} L^+(G^{ad}), \quad (3.14)$$

$$L(\bar{G}) = \bigcup_{\hat{w} \in W_Q} N^-(\bar{G}) \hat{w} L^+(\bar{G}). \quad (3.15)$$

Consider the twisted loop group [33] with the Lie algebra $\mathcal{L}_0(\mathfrak{g}, t, \sigma_{r,s})$ (3.9). It is the subgroup of the equivariant maps

$$L_0(G, \sigma_{r,s}) = \{g(t) \mid \sigma_{r,s}(g(t)) = g(\omega t)\}. \quad (3.16)$$

Let us look on its decompositions (3.14), (3.15)

$$L_0(G, \sigma_{r,s}) = \{g(t) = g_-(t) t^\gamma g_+(t)\}, \quad \begin{cases} 1. \gamma \in \tilde{Q}^\vee, (C.31) & \text{for } G = \bar{G}, \\ 2. \gamma \in \tilde{P}^\vee, (C.30) & \text{for } G = G^{ad}. \end{cases} \quad (3.17)$$

From (3.16) we find that $\sigma_{r,s}(g_\pm(t)) = g_\pm(\omega t)$ and

$$\omega^\gamma t^\gamma = t^{\sigma_{r,s}(\gamma)}. \quad (3.18)$$

From this condition we find that γ are not arbitrary, but satisfies the following conditions

$$(\omega^\gamma = 1, t^\gamma = t^{\sigma_{r,s}(\gamma)}) \leftrightarrow \begin{cases} 1. \gamma \in p\tilde{Q}^\vee & \text{for } G = \bar{G}, \\ 2. \gamma \in p\tilde{P}^\vee & \text{for } G = G^{ad}, \end{cases} \quad (p = \text{ord } \sigma_{r,s}), \quad (3.19)$$

where \tilde{P}^\vee and \tilde{Q}^\vee are invariant coweight and coroot lattices (C.30), (C.31).

4 Holomorphic bundles over elliptic curves and automorphisms of Lie algebras

We construct holomorphic bundles over elliptic curves as in Section 2.1 passing to the Jacobi description of the elliptic curves as the quotient of \mathbb{C}^* . Let $t = \exp 2\pi i z/p = \mathbf{e}(z/p) \in \mathbb{C}^*$, where $p = \text{ord } \sigma$, be the parameter of the loop groups and algebras considered in previous Section. The action on z of the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$ in terms of t assumes the form

$$\begin{aligned} z &\rightarrow z + 1 & t &\rightarrow t\omega, \quad (\omega = \mathbf{e}(1/p)), \\ z &\rightarrow z + \tau & t &\rightarrow t\tilde{q}, \quad (\tilde{q} = \mathbf{e}(\tilde{\tau}), \tilde{\tau} = \tau/p). \end{aligned}$$

In the coordinate t the curve Σ_τ is defined as the quotient $C_{\tilde{q}} = \mathbb{C}^*/\mathbb{Z}\tilde{q}$. We identify $C_{\tilde{q}}$ with the annulus

$$C_{\tilde{q}} = \{t \in \mathbb{C}^* \mid |\tilde{q}^{\frac{1}{2}}| \leq |t| < |\tilde{q}^{-\frac{1}{2}}|\}. \quad (4.1)$$

4.1 Adjoint bundles

Let \mathcal{P} be a principle G -bundle over $C_{\tilde{q}}$. Consider the bundle $\mathcal{A}d_G^\sigma = \mathcal{P} \otimes_{G^{ad}} \mathfrak{g}$ over $C_{\tilde{q}}$. Its sections $\Gamma(\mathcal{A}d_G^\sigma) = \{\Psi(t)\}$ are the equivariant holomorphic maps $C^* \rightarrow \mathfrak{g}$ with respect to the action of finite order automorphism $\sigma_{r,s}$ (3.9).

$$\Psi(t\omega) = \sigma_{r,s}(\Psi(t)), \quad \omega^p = 1. \quad (4.2)$$

The transition operator corresponding to another cycle is

$$\Psi(\tilde{q}t) = R(t)\Psi(t), \quad (4.3)$$

where the $R(t)$ action is the adjoint action, $R(t) \in L(G^{ad})$ (3.14) for $\hat{w} = 1$.

The consistency condition of these transformations implies

$$\sigma_{r,s}(R(t)) = \zeta R(t\omega)\sigma_{r,s}, \quad \zeta \in \mathcal{Z}(\bar{G}). \quad (4.4)$$

If $\zeta = Id$, then (4.4) means that $R(t) \in L_0(G^{ad}, \sigma)$ (3.16).

The gauge group $\mathcal{G}_\sigma(G)$ is the group of automorphisms of the $Ad_G^\sigma(C_{\bar{q}})$ -bundle. It follows from (4.2) that it is the group of equivariant holomorphic maps (3.16). Remind that in the decomposition (3.17) γ is not arbitrary but satisfies (3.19.2).

The group $\mathcal{G}_\sigma(G)$ acts on the transition operators as

$$R(t) \rightarrow f(\tilde{q}t)R(t)f^{-1}(t). \quad (4.5)$$

This action preserves (4.4).

We attach to the marked point $t = 1$ a flag variety $Flag \sim G/P$, where P is a parabolic subgroup of G and assume that $\mathcal{G}_\sigma(G)$ preserves the flag. Then the gauge group is defined as

$$\mathcal{G}_\sigma(G) = \{f(t) : C^* \rightarrow G^{ad}, f \in L_0(G^{ad}, \sigma), f(t)|_{t=1} = P\}. \quad (4.6)$$

Lemma 4.1 *The bundle Ad_G^σ defined by (4.2), (4.3) is gauge equivalent to the bundle Ad_G^ν , where ν is the diagram automorphism. Ad_G^ν is defined by the transitions of its sections*

$$\tilde{\Psi}(t\omega) = \nu(\tilde{\Psi}(t)), \quad \omega = \mathbf{e}(1/r), \quad (4.7)$$

$$\tilde{\Psi}(\tilde{q}t) = \tilde{R}(t)\tilde{\Psi}(t), \quad \tilde{R}(t) = Ad_{t^{-p\kappa}}R(t), \quad (4.8)$$

where $p = \text{ord } \sigma$ and $\mathbf{e}(\kappa)$ is (3.5).

Proof

Consider the gauge transformation $f(t) = t^x$, where $x \in \mathfrak{h}_0$. It follows from (4.2) that

$$\sigma_{r,s} \xrightarrow{f} \omega^x t^x \sigma_{r,s} t^{-x} = \omega^x \sigma_{r,s}.$$

If $x = -p\kappa$, where κ is defined in (3.5), then under this transformations $\sigma_{r,s} \xrightarrow{f} \nu$ and we replace (4.2) on (4.7).

In this way we stay with the diagram automorphism (4.7). Acting by

$$t^{-p\kappa} = t^{r \sum_{j,k=1}^n (a^{-1})_{jk} s_k H_j}, \quad \left(\kappa = \frac{r}{p} \sum_{j,k=1}^n (a^{-1})_{jk} s_k H_j \right) \quad (4.9)$$

on $R(t)$ we come to (4.8). \square

Remark 4.1 *Consider the monodromy $\mathcal{M} = \mathbf{e}(-r \sum_{j,k=1}^n (a^{-1})_{jk} s_k H_j)$ of the gauge transform $t^{-p\kappa}$. The inverse Cartan matrices $(a_{jk})^{-1}$ for Dynkin diagrams with nontrivial symmetries have fractional matrix elements $(a_{jk})^{-1}$ such that $(a_{jk}^l)^{-1}$ are integer, where l is the order of the center $\mathcal{Z}(\bar{G})$ (2.7) of \bar{G} . Since the center is isomorphic to the quotient of the weight and the root lattices $\mathcal{Z}(\bar{G}) \sim P/Q$, the monodromy \mathcal{M} is the element of $\mathcal{Z}(\bar{G})$. Therefore, \mathcal{M} is well defined as an element of the Cartan subgroup $H^{ad} \subset G^{ad}$.*

Remark 4.2 It follows from (4.7) that the sections $\tilde{\Psi} \in \Gamma(\mathcal{A}d_G^\nu)$ have the periodicities

$$\tilde{\Psi}(t\omega^r) = \tilde{\Psi}(t), \quad (\omega = \mathbf{e}(1/p)), \quad (4.10)$$

while the sections $\Psi \in \Gamma(\mathcal{A}d_G^\sigma)$ have the periodicities

$$\Psi(t\omega^p) = \Psi(t), \quad (p \geq r).$$

Now we consider the bundles $\mathcal{A}d_G^\nu(C_q)$ defined by (4.7), (4.8). The condition (4.4) assume the form

$$\nu(R(t)) = \zeta R(t\omega), \quad \zeta \in \mathcal{Z}(\bar{G}), \quad \omega = \mathbf{e}(1/r). \quad (4.11)$$

The moduli space of the bundles for fixed ν and ζ is defined as the set of $\{R(t)\}$ satisfying (4.11) up the gauge transformations (4.5)

$$Bun_{\nu, \zeta}(\mathcal{A}d_G^\nu(C_{\bar{q}})) = \{R(t) \mid \nu(R(t)) = \zeta R(te^{2\pi i/r})\} / \mathcal{G}_\nu(G). \quad (4.12)$$

Here $\mathcal{G}_\nu(G)$ is the group (4.6). Our goal is to describe a big cell in this space. First we prove that the bundles $\mathcal{A}d_G^\nu(C_q)$ are topologically trivial.

We will prove below (Proposition 4.2) that any transition operator $R(t)$ in a neighborhood of constant maps from $C_{\bar{q}}$ to the Cartan subgroup $\mathbf{x} = \mathbf{e}(\mathbf{u}) \in \mathcal{H}(G)$ is represented by the gauge transformation $f(t) = 1 + \mathfrak{x}(t)$

$$R(t) = f^{-1}(qt)\mathbf{x}f(t). \quad (4.13)$$

Passing to the Cartan subalgebra we rewrite (4.11) as

$$\nu(\mathbf{u}) - \mathbf{u} = \xi, \quad (\zeta = \mathbf{e}(\xi)), \quad (4.14)$$

Since $R(t)$ takes values in G^{ad} the element \mathbf{u} is defined up to the shift on the weight lattice

$$\mathbf{u} \sim \mathbf{u} + \gamma, \quad \gamma \in P. \quad (4.15)$$

On the other hand, $\xi \in P$ in (4.14) is defined up to addition an element from the root lattice $\beta \in Q$. Note that ξ is not an arbitrary element of P but belongs to the image of the operator Υ acting on \mathfrak{h} as

$$\Upsilon : \mathfrak{h} \rightarrow \mathfrak{h}, \quad x \mapsto \nu(x) - x. \quad (4.16)$$

In particular, it acts on the weight lattice $\Upsilon : P \rightarrow P$. Let

$$P^\Upsilon = im(\Upsilon), \quad P^\Upsilon = \{\xi \in P \mid \xi = \nu(\gamma) - \gamma, \gamma \in P\}. \quad (4.17)$$

Proposition 4.1 The bundles $\mathcal{A}d_G^\nu(C_q)$ are topologically trivial.

Proof

We prove that any $\xi \in P^\Upsilon$ (4.17) can be gauged away by the shift of \mathbf{u} (4.15) by an element $\gamma \in P$

$$(\xi \in im(\Upsilon)) \Rightarrow (\xi = \nu(\gamma) - \gamma, \gamma \in P).$$

Represent $x \in im(\Upsilon)$ in the basis Ξ of fundamental weights (C.28) $x = c_1\varpi_1 + \dots + c_n\varpi_n$. Since $\nu(\Xi) = \Xi$ then Ξ is an union of the orbits

$$\mathcal{O}_j = \{\varpi_j, \nu(\varpi_j), \dots, \nu^{r-1}(\varpi_j)\}.$$

of length $r = 1, 2$ or 3 (for D_4). The operator Υ acts on orbit \mathcal{O} of the length $r > 1$ as the r -order matrix

$$\begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & -1 & 1 \\ 1 & 0 & \dots & 0 & -1 \end{pmatrix},$$

This operator can be inverted on the $r - 1$ -dimensional subspaces

$$\{c_j \varpi_j + \dots + c_{\nu^{r-1}j} \nu^{r-1}(\varpi_j) \mid c_j + \dots + c_{\nu^{r-1}j} = 0\}. \quad (4.18)$$

In terms of the ν -action this condition can be reformulated as

$$\{x \in \text{im}(\Upsilon)\} \Leftrightarrow \{x + \nu(x) + \dots + \nu^{r-1}(x) = 0\}. \quad (4.19)$$

For $\xi \in P$ all coefficients $c_j = n_j$ are integer, and $\xi \in \text{im}(\Upsilon)$ if $n_j + \dots + n_{\nu^{r-1}j} = 0$. In this case ξ in (4.14) can be adsorbed by the transformation $\mathbf{u} \rightarrow \mathbf{u} + \gamma$, where $\gamma = \sum_j n_j \varpi_j$ and $\sum_j n_j = 0$. ■

The last statement means that the adjoint bundles of Lie algebras defined by (4.2) has only one component corresponding to the trivial characteristic class and in this way the bundles are topologically trivial. Thus, instead (4.14) we can consider

$$\nu(\mathbf{u}) - \mathbf{u} = 0. \quad (4.20)$$

It means that $\mathbf{u} \in \mathfrak{h}_0$.

Now we prove that in a neighborhood of \mathbf{u} almost all transitions $R(t)$ are images of $\mathbf{e}(\mathbf{u})$ under small gauge transforms.

Proposition 4.2 *Let $\mathbf{x} = \mathbf{e}(\mathbf{u})$, $\mathbf{u} \in \mathfrak{h}_0$ (C.13). Then there exists a small gauge transformation $(f(t) = 1 + \mathfrak{r}(t))$, $\mathfrak{r}(t) \in \text{Lie}(\mathcal{G}_\nu(G))$ such that near the Cartan element \mathbf{x} almost all $R(t)$ can be represented as a gauge transform of \mathbf{x}*

$$R(t) = (1 - \mathfrak{r}(tq))\mathbf{x}(1 + \mathfrak{r}(t)). \quad (4.21)$$

Proof

Represent $R(t)$ as $R(t) = \mathbf{e}(\mathbf{u}) + \delta R(t)$, where $\delta R(t)$ is small. Then infinitesimally (4.21) can be rewritten as

$$\mathfrak{r}(t\tilde{q})\mathbf{e}(\mathbf{u}) - \mathbf{e}(\mathbf{u})\mathfrak{r}(t) = \delta R(t) - \delta R_0. \quad (4.22)$$

Here δR_0 is a constant element. The subtraction of δR_0 provides the r.h.s. of (4.22) belongs to the image of the operator

$$D_{\tilde{q}}(\mathbf{u}) : \text{Lie}(\mathcal{G}_\nu(G)) \rightarrow \text{Lie}(\mathcal{G}_\nu(G)) \otimes \Omega^{(1,0)}(C_{\tilde{q}}),$$

$$(D_{\tilde{q}}(\mathbf{u})\mathfrak{r})(t) = (Ad_{\mathbf{e}(\mathbf{u})}\mathfrak{r}(t\tilde{q}) - \mathfrak{r}(t)) dt/t.$$

The prove is based on the existence of the Green function $G(\mathbf{u}, t)$ of $D_{\tilde{q}}$ taking values in $\mathfrak{g} \otimes \mathfrak{g}$

$$G(\mathbf{u}, t\tilde{q}, t_1)\mathbf{e}(\mathbf{u}) - \mathbf{e}(\mathbf{u})G(\mathbf{u}, t, t_1) = \mathcal{I} \cdot \delta(t, t_1) - \mathcal{I}^0. \quad (4.23)$$

Here conjugations by $\mathbf{e}(\mathbf{u})$ is taken with respect to the first argument, $\delta(t, t_1)$ is a linear functional on the space of Laurent polynomials $\mathbb{C}[t^{-1}, t]$ (B.5), (B.6) and \mathcal{I} and \mathcal{I}^0 are the two-tensor on

$\mathfrak{g} \otimes \mathfrak{g}$, defined in the following way. Let $\{T_a\}$ be the basis (C.16) and $c_{a,b} = (T_a, T_b)$ is the Killing metric (C.17), (C.26) in this basis. Then

$$\mathcal{I} = \sum_{a,b} c^{a,b} T_a \otimes T_b, \quad \mathcal{I}^0 = \sum_{k=1}^l \mathfrak{e}_k^0 \otimes \mathfrak{e}_k^0, \quad (\text{see (C.15), (C.25)}).$$

We have $(\mathcal{I}, T_c)_2 = T_c \otimes Id$, where $(\mathcal{I}, T_c)_2 = \text{tr}_2(\mathcal{I}, T_c)$. In fact, $G(\mathbf{u}, t, t_1)$ is defined in terms of the classical r -matrix calculated below (6.3), (6.4)

$$G(\mathbf{u}, t, t_1) = r(\mathbf{u}, t/t_1).$$

It satisfies (4.23) due to (6.3), (6.4), (4.58).

Solutions of (4.22) in terms of $G(t, s)$ assume the form

$$\mathfrak{x}(\mathbf{u}, t) = \oint_{|t_1|=|t|(1+\varepsilon)} (G(\mathbf{u}, t, t_1) X(t_1))_2, \quad (\varepsilon > 0).$$

where $X(s) = \delta R(s) - \delta R_0$. ■

Thus, we found a gauge transformation

$$R(t) \xrightarrow{f(t)} \mathbf{x} \in \mathcal{H}_0^{ad} \subset G_0^{ad}, \quad \mathbf{x} = \mathbf{e}(\mathbf{u}). \quad (4.24)$$

where the element $\mathbf{u} \in \mathfrak{h}_0$.

We analyze solutions of (4.20) up to the transformations preserving its form (the residual gauge transformations). Such solutions define the moduli space $Bun_{\nu, \zeta}(\mathcal{A}d_G^\nu(C_q))$ (4.12) for $\zeta = Id$.

First we described the residual gauge transformations. Let W_0 be the Weyl subgroup of \mathfrak{g}_0 defined in Appendix C. It is the Weyl group generated by the reflections of the root system \tilde{R} , and let \tilde{P}^\vee is the coweight lattice of \mathfrak{g}_0 (C.30).

Proposition 4.3 *The equation (4.20) is invariant with respect the following transformations*

$$\mathbf{u} \rightarrow w \cdot (\mathbf{u}) + \gamma_1 + \gamma_2 \tau, \quad \gamma_1 \in \tilde{P}^\vee, \quad \gamma_2 \in \tilde{P}^\vee, \quad w \in W_0, \quad (4.25)$$

Proof

Remind that the elements $\exp 2\pi i \gamma$ for $\gamma \in \tilde{P}^\vee$ generates the center of the universal covering group \tilde{G}_0 . In other words, $\exp 2\pi i \gamma$ is the unity element in \mathcal{H}_0^{ad} . Therefore, $\Re \mathbf{u}$ in (4.24) is an element of the quotient

$$\Re \mathbf{u} \in \mathfrak{h}_0^{\mathbb{R}} / \tilde{P}^\vee, \quad (\mathbf{u} + \gamma \sim \mathbf{u}, \text{ for } \gamma \in \tilde{P}^\vee). \quad (4.26)$$

In this way $\Re \mathbf{u}$ can be chosen from the fundamental domain in $\mathfrak{h}_0^{\mathbb{R}}$ under the action of the affine Weyl group $\tilde{W}'_{aff} = W_0 \ltimes \tilde{P}^\vee$ (C.37).

The residual gauge transformations of \mathbf{u} is the subgroup $L_0(\mathcal{H}^{ad}) \subset \mathcal{G}_\nu(C_q, G^{ad})$ (4.6) of the equivariant maps to \mathcal{H}^{ad} (see (3.19))

$$L_0(\mathcal{H}^{ad}) = \{f(t) = t^\gamma h(t) \mid \gamma \in r\tilde{P}^\vee, \quad h(t) = \sum_k h_k t^k, \quad h_k \in \mathcal{H}^{ad}, \quad \nu(h_k) = \omega^{-k} h_k\}. \quad (4.27)$$

It acts on \mathbf{x} (4.24) and on \mathbf{u} as

$$\mathbf{x} \rightarrow \tilde{q}^{r\gamma} \mathbf{x} = q^\gamma \mathbf{x}, \quad (\tilde{q} = \mathbf{e}(\tau/r), \quad q = \mathbf{e}(\tau)), \quad \mathbf{u} \rightarrow \mathbf{u} + \gamma\tau, \quad \gamma \in \tilde{P}^\vee. \quad (4.28)$$

Then due to (4.26) and (4.28) solutions of (4.20) are invariant under the shifts

$$\mathbf{u} \rightarrow \mathbf{u} + \gamma_1 + \gamma_2\tau, \quad (\gamma_1, \gamma_2 \in \tilde{P}^\vee). \quad (4.29)$$

In this way, \mathbf{u} belongs to the fundamental domain

$$\mathbf{u} \in \mathfrak{h}_0/\tilde{W}'_{BS}, \quad (4.30)$$

where \tilde{W}'_{BS} is the Bernstein-Schwarzman group [3] generated by W_0 and the shifts (4.29)

$$\tilde{W}'_{BS} = W_0 \ltimes (\tau\tilde{P}^\vee \oplus \tilde{P}^\vee). \quad (4.31)$$

■

Thus the moduli space $Bun_{\nu, Id}(\mathcal{A}d_G^\nu(C_{\tilde{q}}))$ (4.12):

- is independent on ζ (4.12);
- is isomorphic to the fundamental domain of the Bernstein-Schwarzman group (4.31)

$$\boxed{Bun_{\nu, Id}(\mathcal{A}d_G^\nu(C_{\tilde{q}})) \sim \mathfrak{h}_0/\tilde{W}'_{BS}(r)}. \quad (4.32)$$

It can be proved that this quotient can be described in terms of the curve C_q and the invariant weight sublattice \tilde{P}^\vee

$$\mathfrak{h}_0/\tilde{W}'_{BS}(r) \sim (C_q \otimes \tilde{P}^\vee)/W_0, \quad q = \mathbf{e}(\tau). \quad (4.33)$$

To pass from the $\mathcal{A}d_G^\nu(C_{\tilde{q}})$ -bundles to $\mathcal{A}d_G^\sigma(C_{\tilde{q}})$ -bundles we act on the sections $\Gamma(\mathcal{A}d_G^\nu(C_{\tilde{q}}))$ by the gauge transform $t^{p\kappa}$ (4.9).

4.2 Holomorphic bundle with non-diagonalized automorphisms

Here we consider a holomorphic bundle (4.2), (4.3), where $R(t)$ is not an element of the Cartan subgroup $\mathcal{H}(G)$ as was assumed in (4.13) and Proposition 4.2, but belongs to the Weyl group $W(G)$. More concretely, let $G = \mathrm{SL}(N, \mathbb{C})$, $N = 2n$ and ν is the outer automorphism of the Lie algebra $\mathfrak{sl}(N, \mathbb{C})$ (C.1) leading to the invariant subalgebra $\mathfrak{sp}(n)$. The action of $R(t)$ is generated by the automorphism λ of the extended Dynkin diagram (C.9) $\tilde{\Pi}^{ext}$ of the root system $A_{2n-1}^{(1)}$. It acts by the cyclic permutation on the roots $\alpha_0 \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_0$, and $R(t) = \Lambda = \lambda^n$. Represent $x \in \mathfrak{sl}(N, \mathbb{C})$ as the $2n \times 2n$ matrix

$$x = \begin{pmatrix} A & B \\ D & C \end{pmatrix} \in \mathfrak{sl}(N, \mathbb{C}), \quad (4.34)$$

Define the action of the outer automorphism ν as

$$\nu(x) = \begin{pmatrix} -\tilde{D} & \tilde{B} \\ \tilde{C} & -\tilde{A} \end{pmatrix}, \quad \tilde{A} = jA^T j, \quad j = \{E_{jk} = \delta_{j, n-k+1}\}. \quad (4.35)$$

The action of λ is performed by the conjugation by Λ

$$\Lambda = \begin{pmatrix} 0 & Id_n \\ Id_n & 0 \end{pmatrix}. \quad (4.36)$$

Note that the transformations ν and λ anti-commute

$$\nu(\Lambda)\Lambda^{-1} = -Id_N. \quad (4.37)$$

4.2.1 Special basis in $\mathfrak{sl}(N, \mathbb{C})$

Since $\nu^2 = Id$ and $\Lambda^2 = Id$ and due to (4.37) the Lie algebra $\mathfrak{sl}(N, \mathbb{C})$ (and more generally $\mathfrak{gl}(N, \mathbb{C})$) acquires $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(a, b)\}$ gradation

$$Ad_\Lambda x = \pm x, \quad \nu(x) = \pm x,$$

(a, b)	(Λ, ν)
(0,0)	(+, +)
(0,1)	(+, -)
(1,0)	(-, +)
(1,1)	(-, -)

(4.38)

Instead of the ν -consistent basis (C.10), (C.14) in $\mathfrak{sl}(N, \mathbb{C})$ we consider a basis related to the gradation (4.38) and the structure (4.34). Consider the orbits in the space of the $\mathfrak{gl}(N, \mathbb{C})$ -basis $\{E_{JK}\}$ under the action of the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ group generated by ν and Ad_Λ . Let \mathcal{O}_{jk} be the orbit passing through E_{jk} , or $E_{j,n+k}$ ($j, k = 1, \dots, n$). Then

$$\{E_{J,K}, (J, K = 1, \dots, N)\} = \cup_{j+k \leq n+1} \mathcal{O}_{jk}.$$

$$\sharp(\mathcal{O}_{jk}) = \begin{cases} 4, & j+k < n+1, \\ 2, & j+k = n+1, \end{cases}$$

We introduce in $\mathfrak{gl}(N, \mathbb{C})$ the basis

$$\{\mathcal{T}_{(a,b),jk}^\alpha, \alpha, a, b = 0, 1, j, k = 1, \dots, n\} \quad (4.39)$$

in terms of the tensor product of the sigma-matrices σ_k , ($k = 0, \dots, 3$) with the $\mathfrak{gl}(n)$ matrices. Let E_{jk} be the basis in $\mathfrak{gl}(n)$. then $\mathcal{T}_{(a,b)}^\alpha$ is defined as follows. For $j+k < n+1$ it takes the form

$$\begin{aligned} \mathcal{T}_{(0,a),jk}^0 &= \sigma_0 \otimes (E_{jk} - (-1)^a E_{n-k+1, n-j+1}), & \mathcal{T}_{(0,a),jk}^1 &= \sigma_1 \otimes (E_{jk} + (-1)^a E_{n-k+1, n-j+1}), \\ \mathcal{T}_{(1,a),jk}^0 &= \sigma_3 \otimes (E_{jk} - (-1)^a E_{n-k+1, n-j+1}), & \mathcal{T}_{(1,a),jk}^1 &= \sigma_2 \otimes (E_{jk} + (-1)^a E_{n-k+1, n-j+1}). \end{aligned} \quad (4.40)$$

If $j+k = n+1$ then

$$\begin{aligned} \mathcal{T}_{(0,0),(j,n-j+1)}^1 &= 2\sigma_1 \otimes E_{j,n-j+1}, & \mathcal{T}_{(0,1),(j,n-j+1)}^0 &= 2\sigma_0 \otimes E_{j,n-j+1}, \\ \mathcal{T}_{(1,0),(j,n-j+1)}^1 &= 2\sigma_2 \otimes E_{j,n-j+1}, & \mathcal{T}_{(1,1),(j,n-j+1)}^0 &= 2\sigma_3 \otimes E_{j,n-j+1}. \end{aligned} \quad (4.41)$$

It will be convenient to select the diagonal elements of the basis. Let $e_j = E_{jj}$. Then for $n = 2l$

$$\begin{aligned} \mathcal{H}_{(0,a),j} &= \mathcal{T}_{(0,a),jj}^0 = \sigma_0 \otimes (e_j - (-1)^a e_{n-j+1}), \\ \mathcal{H}_{(1,a),j} &= \mathcal{T}_{(1,a),jj}^0 = \sigma_3 \otimes (e_j - (-1)^a e_{n-j+1}), \end{aligned} \quad 0 \leq j \leq l. \quad (4.42)$$

For $n = 2l+1$

$$\begin{aligned} \mathcal{H}_{(0,a),j} &= \mathcal{T}_{(0,a),jj}^0 = \sigma_0 \otimes (e_j - (-1)^a e_{n-j+1} + \delta_{a,1} e_{l+1}), \\ \mathcal{H}_{(1,a),j} &= \mathcal{T}_{(1,a),jj}^0 = \sigma_3 \otimes (e_j - (-1)^a e_{n-j+1} + \delta_{a,0} e_{l+1}), \end{aligned} \quad 0 \leq j \leq l, \quad (4.43)$$

$$\begin{aligned} \mathcal{H}_{(0,1),l+1} &= \mathcal{T}_{(0,a),l+1,l+1}^0 = \sigma_0 \otimes e_{l+1}, \\ \mathcal{H}_{(1,0),l+1} &= \mathcal{T}_{(1,a),l+1,l+1}^0 = \sigma_3 \otimes e_{l+1}. \end{aligned}$$

Consider the action of ν and λ on the basis. From (4.35) we find

$$\nu(\mathcal{T}_{(a,b)}^\alpha) = (-1)^b \mathcal{T}_{(a,b)}^\alpha, \quad (4.44)$$

It follows from (4.36) that the adjoint action of Λ takes the form $Ad_\Lambda(\mathcal{T}_{(a,b)}^\alpha) = Ad_{\sigma_1 \otimes Id_n}(\mathcal{T}_{(a,b)}^\alpha)$. Therefore

$$Ad_\Lambda(\mathcal{T}_{(a,b)}^\alpha) = (-1)^a \mathcal{T}_{(a,b)}^\alpha. \quad (4.45)$$

The Killing form $Tr(E_{JK}E_{IM}) = \delta_{KI}\delta_{JM}$ in $\mathfrak{gl}(N, \mathbb{C})$ leads to the following norm for the generators

$$\langle \mathcal{T}_{(a,b),jk}^\alpha, \mathcal{T}_{(c,d),im}^\beta \rangle = 4\delta^{\alpha\beta} \delta_{ac} \delta_{bd} \delta_{ki} \delta_{jm}. \quad (4.46)$$

In what follows we need the commutation relations of $\mathcal{H}_{(0,0),i}$ with other generators. For $n = 2l + 1$ and $j \leq k$

$$[\mathcal{H}_{(0,0),i}, \mathcal{T}_{(a,b),jk}^\alpha] = \begin{cases} 1. & (\delta_{ij} - \delta_{ik}) \mathcal{T}_{(a,b),jk}^\alpha, & j, k \leq l, \\ 2. & \delta_{ij} \mathcal{T}_{(a,b),jk}^\alpha, & j \leq l, k = l + 1, \\ 3. & (\delta_{ij} + \delta_{ik}) \mathcal{T}_{(a,b),jk}^\alpha, & j \leq l, k > l + 1, \\ 4. & \delta_{i,n-k+1} \mathcal{T}_{(a,b),jk}^\alpha, & j = l + 1, k \geq l + 1, \\ 5. & (\delta_{ij} - \delta_{ik}) \mathcal{T}_{(a,b),jk}^\alpha, & j, k > l + 1. \end{cases} \quad (4.47)$$

For $j \geq k$ the right hand side changes sign. If $n = 2l$ then the lines 2. and 4. omit.

4.2.2 Bundle $Ad_{SL}^{\nu,\Lambda}(C_{\tilde{q}})$

The bundle $Ad_{SL}^{\nu,\Lambda}(C_q)$ is defined by the transitions acting on its sections $\Gamma(Ad_{SL}^{\nu,\Lambda})$ as

$$\Psi(-t) = \nu(\Psi(t)), \quad (4.48)$$

$$\Psi(\tilde{q}t) = Ad_\Lambda \Psi(t) \quad (4.49)$$

(compare with (4.2), (4.3)). These conditions are consistent because of (4.37). In particular, (4.37) means that the bundle $Ad_{SL}^{\nu,\Lambda}(C_q)$ defined by (4.48), (4.49) is topologically non-trivial (see (4.4)).

The ν -invariant subgroup of $SL(N, \mathbb{C})$ is the symplectic group $Sp(n, \mathbb{C})$. Its subgroup

$$G_0 = \{f \in Sp(n, \mathbb{C}) \mid Ad_\Lambda f = f\}$$

preserves (4.37). It plays the role of the constant gauge transformations. Consider the diagonal gauge transformations. Let $\vec{u} = (u_1, \dots, u_l)$, where $l = [n/2]$ and $\tilde{\vec{u}} = (u_l, \dots, u_1)$. Define the diagonal matrix $\mathbf{u} \in \mathcal{H}_{00} \subset Lie(G_0)$ (4.42), (4.43)

$$\mathbf{u} = \begin{cases} \text{diag}(\vec{u}, -\tilde{\vec{u}}, \vec{u}, -\tilde{\vec{u}}), & n = 2l, \\ \text{diag}(\vec{u}, 0, -\tilde{\vec{u}}, \vec{u}, 0, -\tilde{\vec{u}}), & n = 2l + 1. \end{cases} \quad (4.50)$$

Since $\mathbf{u} \in \mathcal{H}_{00}$ we can replace (4.49) by the more general condition

$$\Psi(\tilde{q}t) = Ad_{e(\mathbf{u})\Lambda} \Psi(t). \quad (4.51)$$

It does not break the consistency condition (4.37). The vector \vec{u} is the tangent vector to the moduli space of the bundle $Bun(Ad_{SL}^{\nu,\Lambda}(C_{\tilde{q}}))$. As above, \vec{u} is defined up to the action of. Let W_0^Λ is the Λ -invariant subgroup of the Weyl group $W(\mathfrak{sp}(n, \mathbb{C}))$, and P_0^\vee is Λ -invariant coweight

lattice of $\mathfrak{sp}(n, \mathbb{C})$. The lattice P_0^\vee has the rank $l = \lfloor \frac{n}{2} \rfloor$. In terms of the basic vectors in $\mathfrak{h} \subset \mathfrak{sl}(N, \mathbb{C})$ its generators have the form

$$\varpi_1^\vee = (e_1 - e_n + e_{n+1} - e_{2n}), \dots, \varpi_l^\vee = \sum_{k=1}^l (e_k - e_{n-k+1} + e_{n+k} - e_{2n-k+1}).$$

The Bernstein-Schwarzman group is defined as

$$W_{BS}^0(r) = W_0^\Lambda \ltimes (\tau P_0^\vee \oplus P_0^\vee). \quad (4.52)$$

In particular the shifts act on \mathbf{u} (4.50) as

$$\mathbf{u} \sim \mathbf{u} + \gamma_1 + \gamma_2 \tau, \quad \gamma_1, \gamma_2 \in P_0^\vee.$$

In this way the moduli space of $\mathcal{A}d_{SL}^{\nu, \Lambda}(C_{\tilde{q}})$ is defined as

$$\boxed{Bun(\mathcal{A}d_{SL}^{\nu, \Lambda}(C_{\tilde{q}})) \sim \mathcal{H}_{00}/W_{BS}^0}. \quad (4.53)$$

Represent the sections of the bundle $\mathcal{A}d_{SL}^{\nu, \Lambda}(C_q)$ in the form

$$\Psi(t) = \sum_{\alpha, a, b=0,1} \psi_{a,b}^\alpha(t) + \xi_{a,b}(t). \quad (4.54)$$

$$\psi_{a,b}^\alpha(t) = \sum_{j,k=1}^l \psi_{(a,b),jk}^\alpha(t) \mathcal{T}_{(ab),jk}^\alpha, \quad \xi_{a,b}(t) = \sum_{j=1}^l \xi_{(a,b),j}(t) \mathcal{H}_{(a,b),j}. \quad (4.55)$$

The section takes value in $\mathfrak{sl}(N, \mathbb{C})$ if

$$\begin{aligned} \sum_{j=1}^l \xi_{(01),j}(t) &= 0, & \text{for } n = 2l, \\ 2 \sum_{j=1}^l \xi_{(01),j}(t) + \xi_{(01),l+1}(t) &= 0, & \text{for } n = 2l + 1. \end{aligned} \quad (4.56)$$

The quasi-periodicity properties of the sections in this basis follow from (4.44), (4.45), (4.47)

$$\psi_{a,b}^\alpha(-t) = (-1)^b \psi_{a,b}^\alpha(t), \quad \xi_{a,b}(-t) = (-1)^b \xi_{a,b}(t), \quad \xi_{a,b}(\tilde{q}t) = (-1)^a \xi_{a,b}(t), \quad (4.57)$$

$$\psi_{(a,b),jk}^\alpha(\tilde{q}t) = u_{jk} \psi_{(a,b),jk}^\alpha(t), \quad (4.58)$$

where

$$u_{jk} = \begin{cases} 1. & (u_j - u_k), \quad j, k \leq l, \\ 2. & u_j, \quad j \leq l, k = l+1, \\ 3. & (u_j + u_k), \quad j \leq l, k > l+1, \\ 4. & u_{n-k+1}, \quad j = l+1, k \geq l+1, \\ 5. & (u_j - u_k), \quad j, k > l+1. \end{cases} \quad (4.59)$$

4.3 Twisted vector bundles

To construct vector bundles over $C_{\tilde{q}}$ with non-trivial characteristic classes we replace the adjoint bundle $\mathcal{A}d_G$ (4.2) by the vector bundle E_G defined as follows. Consider a (reducible) representation π of the Lie algebra \mathfrak{g} (C.1) in V . Let V_η be an irreducible \mathfrak{g} -module with the highest

weight $\eta \in P$. Assume that the diagram automorphism ν acts in a such way that $\nu\eta \neq \eta$. The module V is a sum the highest modules

$$V = \oplus_{j=0}^{r-1} V_{\nu^j(\eta)}, \quad \eta \in P. \quad (4.60)$$

For example, we can take

\mathfrak{g}	η	V_η	$\nu(\eta)$	$V_{\nu(\eta)}$
A_{n-1}	ϖ_1	\underline{n}	ϖ_{n-1}	$\underline{\bar{n}}$
D_n	ϖ_n	left spinors	ϖ_{n-1}	right spinors
E_6	ϖ_1	$\underline{27}$	ϖ_6	$\underline{\bar{27}}$
D_4	ϖ_1	$\underline{8}^V$	$\varpi_3, \nu^2(\eta) = \varpi_4$	$\underline{8}^L, \underline{8}^R$

Table 1. Examples of representations corresponding to non-trivial bundles

We define the vector bundle $E_{\bar{G}}$ as the associated bundle $E_{\bar{G}} = \mathcal{E}_{\bar{G}} \times_{\bar{G}} V$. The sections $\psi(t) \in \Gamma(E_{\bar{G}})$ are transformed as (compare with (4.7))

$$\psi(t\omega) = \nu(\psi(t)), \quad (\omega = \mathbf{e}(1/r)), \quad (4.61)$$

$$\psi(\tilde{g}t) = R(t)\psi(t), \quad R \in \pi(g), \quad g \in \bar{G}. \quad (4.62)$$

From (4.61) we find that the action of ν changes components in the decomposition (4.60). If, for example, $\psi(t)$ takes value in $V_{\nu^i(\eta)}$ then $\psi(t\omega)$ takes value in $V_{\nu^{i+1}(\eta)}$.

The gauge transformations of the sections $\Gamma(E_{\bar{G}})$ preserving the consistency condition (4.11) have the same form as before (4.5). Then, locally (4.11) takes the form (4.14).

It is crucial, that in the contrast with the adjoint bundles the transition operators R in the vector bundles act in the modules of the group \bar{G} . It means, in particular, that in (4.14) ξ is defined up to $\gamma \in Q$.

Similar to the lattice P^Υ (4.17) define the lattice

$$Q^\Upsilon = \text{im}(\Upsilon), \quad Q^\Upsilon = \{\xi \in Q \mid \xi = \nu(\gamma) - \gamma, \gamma \in Q\}. \quad (4.63)$$

Consider the r.h.s of (4.14). It is an element of P^Υ (4.17). On the other hand, since $\mathbf{x} = \mathbf{e}(\mathbf{u})$ is an element of the Cartan subgroup $\bar{\mathcal{H}} \subset \bar{G}$ the element $\mathbf{u} \in \mathfrak{h}$ is defined up to elements from Q . The Q -action on \mathbf{u} in (4.14) means that ξ is defined up to elements from Q^Υ . In other words, ξ can be considered as an element from the quotient group

$$\xi \in P^\Upsilon / Q^\Upsilon = \mathcal{Z}^\nu(\bar{G}) \subseteq \mathcal{Z}(\bar{G}) \sim P/Q,$$

The condition (4.18) allows one to describe the quotient explicitly in terms of the basis of fundamental weights. In Appendix A we prove that it has the following structure

	\mathfrak{g}	\mathfrak{g}_0	$\mathcal{Z}^\nu(\bar{G}) = P^\Upsilon/Q^\Upsilon$	$\mathcal{Z}(\bar{G}) = P/Q$	Weights generating $\mathcal{Z}^\nu(\bar{G})$
	1	2	3	4	5
1.	A_{2n-1}	C_n	\mathbb{Z}_n	\mathbb{Z}_{2n}	ϖ_1
2.	A_{2n}	B_n	\mathbb{Z}_{2n+1}	\mathbb{Z}_{2n+1}	ϖ_1
3.	D_{n+1}	B_n	\mathbb{Z}_2	\mathbb{Z}_4 , or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$	ϖ_n
4.	D_4	G_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	ϖ_1, ϖ_3
5.	E_6	F_4	\mathbb{Z}_3	\mathbb{Z}_3	ϖ_1

Table 2. The group $\mathcal{Z}^\nu(\bar{G})$

Thus, for the group $\mathrm{SL}(N, \mathbb{C})$ ($N = 2n$) and Spin_{2n+2} the group $\mathcal{Z}^\nu(\bar{G})$ does not coincides with $\mathcal{Z}(\bar{G})$.

Remark 4.3 The group $\mathcal{Z}^\nu(\bar{G})$ differs from the center $\mathcal{Z}(\bar{G}_0)$. For example, in the first line $C_n \rightarrow \bar{G}_0 = \mathrm{Sp}_n$, $\mathcal{Z}(\mathrm{Sp}_n) = \mathbb{Z}_2$, while $\mathcal{Z}^\nu(\mathrm{SL}_{2n}) = \mathbb{Z}_n$.

Now consider the transformations of \mathbf{u} preserving (4.14). The elements $\exp 2\pi i \gamma$ for $\gamma \in \tilde{Q}^\vee$ are the unity elements in \bar{G}_0 . Therefore, \mathbf{u} in (4.24) is an element of the quotient

$$\Re \mathbf{u} \in \mathfrak{h}_0^\mathbb{R} / \tilde{Q}^\vee, \quad (\mathbf{u} + \gamma \sim \mathbf{u}, \text{ for } \gamma \in \tilde{Q}^\vee). \quad (4.64)$$

Thus, $\Re \mathbf{u}$ can be chosen from the fundamental domain in $\mathfrak{h}_0^\mathbb{R}$ under the action of the affine Weyl group $\tilde{W}_{aff} = W_0 \ltimes \tilde{Q}^\vee$ (C.37).

The residual gauge transformations of \mathbf{u} is the subgroup $L_0(\bar{\mathcal{H}}) \subset \mathcal{G}_\nu(C_q, \bar{G})$ (4.6) of the equivariant maps to $\bar{\mathcal{H}}$ (see (3.17) and (3.19))

$$L_0(\bar{\mathcal{H}}) = \{f(t) = t^\gamma h(t) \mid \gamma \in r\tilde{Q}^\vee, h(t) = \sum_k h_k t^k, h_k \in \bar{\mathcal{H}}, \nu(h_k) = \omega^{-k} h_k\}.$$

Similar to (4.25) the moduli space of the $E_{\bar{G}}$ -bundles is the set of pairs $(\mathbf{u} \in \mathfrak{h}_0, \xi \in P)$ satisfying (4.14), and invariant with respect to the action

$$\mathcal{P}_{E_{\bar{G}}} : \begin{cases} \mathbf{u} \rightarrow \mathbf{u} + \gamma_1 + \gamma_2 \tau, & \gamma_1, \gamma_2 \in \tilde{Q}^\vee, \\ \xi \rightarrow \xi + \gamma_3 & \gamma_3 \in Q^\Upsilon. \end{cases} \quad (4.65)$$

We define the corresponding to the lattice \tilde{Q}^\vee the Bernstein-Schwarzman group

$$\tilde{W}_{BS} = W_0 \ltimes (\tau \tilde{Q}^\vee \oplus \tilde{Q}^\vee). \quad (4.66)$$

Thus, for a fixed $\zeta = \mathbf{e}(\xi) \in \mathcal{Z}^\nu(\bar{G})$ we can identify the moduli space with the fundamental domain of the Bernstein-Schwarzman group \tilde{W}_{BS} .

$$\boxed{Bun_{\sigma, \zeta}(E_{\bar{G}}(C_{\bar{q}})) \sim \mathfrak{h}_0 / \tilde{W}_{BS}(r).} \quad (4.67)$$

Thus, the general solutions of the equation (4.14) has the form

$$\mathbf{u} + \varpi_j, \quad \mathbf{u} \in Bun_{\sigma, \zeta}(E_{\bar{G}}(C_q)), \quad (4.68)$$

and ϖ_j is a fundamental weight of the Lie algebra \mathfrak{g} satisfying the equation

$$\nu(\varpi_j) - \varpi_j = \xi. \quad (4.69)$$

Solutions of this equation are given in the column 5 of Table 2. The proof is contained in Appendix A.

The element ξ defines the characteristic class of the $E_{\bar{G}}$ -bundle. Note, that the moduli space are independent on the characteristic class ζ .

As above to pass to the general automorphism $\sigma_{r,s} = \nu \mathbf{e}(\kappa)$ (3.5), (3.7) we apply the gauge transformation $f = t^{p\kappa}$ (4.9).

5 Higgs bundles

The Higgs bundle $\mathcal{H}(G, \Sigma_g)$ over the Riemann surface Σ_g is the pair $(E_{\bar{G}}, \Phi)$, where $E_{\bar{G}} = \mathcal{E} \times_G V$ is a holomorphic vector G -bundle, associated with the principal bundle \mathcal{E} and Φ is a Higgs field. It is a holomorphic 1-form on Σ_g taking values in $\text{End } E_{\bar{G}}$. The Higgs bundle can be equipped with a symplectic form. The group of automorphisms of the underlying vector bundle can be lifted to the Higgs field in a such way that it acts as the symplectomorphisms of the Higgs bundle.

In what follows we describe the Higgs bundles related to constructed above three types of holomorphic G -bundles over elliptic curves $\mathcal{A}d_G$, $\mathcal{A}d_{SL}^{\nu, \Lambda}$ and $E_{\bar{G}}$.

5.1 Higgs bundles related to $\mathcal{A}d_G$ and $E_{\bar{G}}$ bundles

The Higgs bundle $\mathcal{H}(G, C_{\tilde{q}})$ is described by the pair $(R(t), \Phi(t))$, where $R(t)$ is the transition operator (4.3) and Φ is a holomorphic one-form in a neighborhood \mathcal{V} of the contour $\gamma_1 = \{|t| = |\tilde{q}^{\frac{1}{2}}|\}$ taking values in the endomorphisms of the bundles $\Phi = \Phi(t) \frac{dt}{t} \in \Omega^{(1,0)}(\mathcal{V}, \text{End}(\mathfrak{g}))$. There is a well defined symplectic form on $\mathcal{H}(G, C_{\tilde{q}})$

$$\Omega = \frac{1}{2\pi i} \oint_{\gamma_1} \langle (D(R^{-1}(t)\Phi(t)) \wedge DR(t)) \rangle. \quad (5.1)$$

We assume that $\Phi(t)$ is the equivariant map $\Phi(t) \in \mathcal{L}_0(\mathfrak{g}, t)$ (3.9).

$$\nu(\Phi(t) = \Phi(t\omega), \quad (\omega = \exp \frac{2\pi i}{r}). \quad (5.2)$$

The form (5.1) is invariant with respect to the $\mathcal{G}_\nu(G)$ -action (4.5) and

$$\Phi(t) \rightarrow f(\tilde{q}t)\Phi(t)f^{-1}(\tilde{q}t). \quad (5.3)$$

This symplectic action leads to the moment map

$$\mu : \mathcal{H}(G, C_{\tilde{q}}) \rightarrow \text{Lie}^*(\mathcal{G}_\sigma(G)),$$

defined by the functional

$$\mathcal{F} = \frac{1}{2\pi i} \oint_{\gamma_1} \langle \Phi(t) \mathfrak{x}(\tilde{q}t) \rangle - \frac{1}{2\pi i} \oint_{\gamma_1} \langle R^{-1}(t)\Phi(t)R(t)\mathfrak{x}(t) \rangle, \quad \mathfrak{x}(t) \in \text{Lie}(\mathcal{G}_\sigma(G)).$$

As in [5] we find that the zero moment map implies that Φ can be extended holomorphically from the boundary $(\gamma_1 = \{|t| = |\tilde{q}^{\frac{1}{2}}|\}, \gamma_2 = \{|t| = |\tilde{q}|^{-\frac{1}{2}}\})$ inside the annulus with at most a single pole at $t = 1$ and such that on γ_1

$$\Phi(\tilde{q}^{-1}t) = R^{-1}(t)\Phi(t)R(t). \quad (5.4)$$

Since the gauge group preserves a flag at $t = 1$ from (5.3) we find that the Higgs field $\Phi(t)$ has a pole at $t = 1$ we assume that the elements of Lie algebra of the gauge transformations

$$\text{Res } \Phi(t)|_{t=1} = \mathbf{S}, \quad (5.5)$$

where S is an element of coadjoint G -orbit attached to the point $t = 1$. Details can be found in [5, 26].

Acting by the gauge transformations on $R(t)$ (4.24) $R(t) \xrightarrow{f(t)} \mathbf{e}(\mathbf{u})$ we transform simultaneously the Higgs field

$$\Phi(t) \xrightarrow{f(t)} L(t).$$

The transformed Higgs field $L(t)$ plays the role of the Lax operator. In these terms (5.4) assumes the form

$$L(\tilde{q}^{-1}t) = \mathbf{e}(-\mathbf{u})L(t)\mathbf{e}(\mathbf{u}). \quad (5.6)$$

Solutions of this equation and (5.5) $L = L(\mathbf{S}, \mathbf{u}, \mathbf{v}, t)$ depends on an additional parameter $\mathbf{v} \in \mathfrak{h}_0^*$.

The reduced phase space can be considered as the cotangent bundle $T^*Bun_\zeta(C_{\tilde{q}}, G)$ to the moduli space of holomorphic bundles with the quasi-parabolic structure at $t = 1$. $(\mathbf{S}, \mathbf{v}, \mathbf{u})$ play the role of local coordinates in $T^*Bun_\zeta(C_{\tilde{q}}, G)$. Here \mathbf{S} is an element of symplectic quotient of a coadjoint orbit of G_0 with respect to the Cartan subgroup action. We come to this point below. The coordinates (\mathbf{v}, \mathbf{u}) are canonical

$$\{v_j, u_k\} = \delta_{jk}. \quad (5.7)$$

and (see (4.67))

$$\mathbf{u} \in \begin{cases} Bun_{\sigma,1}(Ad_G(C_{\tilde{q}})) \sim \mathfrak{h}_0/\tilde{W}_{BS}^{ad} \\ Bun_{\sigma,\zeta}(E_{\tilde{G}}(C_{\tilde{q}})) \sim \mathfrak{h}_0/\tilde{W}_{BS}^{sc} \end{cases} \quad (5.8)$$

Summarizing we define the Lax operator by the conditions We find from (5.2) and (5.4)

$$1. \nu(L(t)) = L(t\omega^{-1}), \quad 2. L(\tilde{q}^{-1}t) = Ad_{\mathbf{e}(\mathbf{u})}L(t), \quad 3. Res L(t) = \mathbf{S}. \quad (5.9)$$

We preserve the same notation \mathbf{S} for the residue of the Lax operator.

Consider the first condition. From (C.2) we have

$$L(t) = \sum_{m=0}^{r-1} L_m(t), \quad (r = 2, 3), \quad (5.10)$$

$$L_m \in \mathfrak{g}_m \leftrightarrow \nu(L_m) = \omega^m L_m, \quad L_m(t\omega) = \omega^{-m} L_m(t). \quad (5.11)$$

To define the Lax operator, satisfying 1. and 2. (5.9) we use the root basis (C.12), (C.13). In the root basis we have

$$L_0(t) = L_{0|\mathfrak{h}}(t)\mathfrak{h}_0 + \sum_{\tilde{\alpha} \in \tilde{R}} L_{0,\tilde{\alpha}}(t),$$

$$L_m(t) = L_{m|\mathfrak{h}}(t)\mathfrak{h}_m + \sum_{\tilde{\alpha} \in R_s} L_{m,\tilde{\alpha}}(t).$$

We decompose the residue 3.(5.9) of L in the basis (C.16)

$$\mathbf{S} = \sum_{m=0}^{r-1} S_m, \quad S_0 = \sum_{k=1}^n S_0^k \mathfrak{e}_k^0 + \sum_{\tilde{\alpha} \in \tilde{R}} S_{\tilde{\alpha}} E_{\tilde{\alpha}}, \quad S_m = \sum_{k=1}^n S_m^k \mathfrak{e}_k^m + \sum_{\tilde{\alpha} \in R_s} S_m^{\tilde{\alpha}} \mathfrak{t}_{\tilde{\alpha}}^m.$$

It is convenient to represent the Cartan part S_0 in the coroot basis (C.27)

$$S_0^{\mathfrak{h}} = \sum_{k=1}^n S_0^k \mathfrak{e}_k^0 = \sum_{\tilde{\alpha} \in \Pi} S_0^{\tilde{\alpha}} H_{\tilde{\alpha}}. \quad (5.12)$$

The invariant component of the Lax operator can be expressed in terms of the functions $g(s, t^r|q)$ (B.1) and $E_1(t^r|q)$ (B.11)

$$L_0(t) = \sum_{k=1}^n (v_k + S_0^k E_1(t^r|q)) \mathfrak{e}_k^0 + \sum_{\tilde{\alpha} \in \tilde{R}} S_{\tilde{\alpha}} g(s_{\tilde{\alpha}}^{-1}(\mathbf{u}), t^r|q) E_{\tilde{\alpha}}, \quad s_{\tilde{\alpha}}(\mathbf{u}) = \exp 2\pi i \mathbf{u}_{\tilde{\alpha}}. \quad (5.13)$$

It satisfies (5.9) due to (B.7), (B.10) and (B.3). The Cartan part $L_0^{\mathfrak{h}}(t)$ can be rewritten in the simple coroot basis (see (5.12))

$$L_0^{\mathfrak{h}}(t) = \sum_{k=1}^n (v_k + S_0^k E_1(t^r|q)) \mathfrak{e}_k^0 = \sum_{k=1}^n v_k \mathfrak{e}_k^0 + \sum_{\tilde{\alpha} \in \tilde{\Pi}} S_{\tilde{\alpha}}^{\tilde{\alpha}} E_1(t^r|q) H_{\tilde{\alpha}}. \quad (5.14)$$

The terms $S_0^k E_1(t|q) \mathfrak{e}_k^0$ in (5.13) violate the residue theorem and, due to (B.12), brake the symmetry 2.(5.9). The origin of this contradiction is that there is residual gauge symmetry. In fact, these terms vanish after the applying the symplectic reduction with respect to the action of the invariant Cartan subgroup \mathcal{H}_0 . The conditions $S_0^k = 0$ play the role of the moment constraints. (See details in [27]). We neglect this term in calculations of the Hamiltonians. The described below r -matrix defines the Poisson brackets before the residual reduction. For this reason it contains the corresponding anomalous term (see (6.3)). Note that L_0 is the standard Lax operator for the spin Calogero-Moser system related to the invariant subalgebra \mathfrak{g}_0 .

The other components of the Lax operator assume the following form. Introduce r functions

$$g^{(m)}(s, t|q) = t^m g(sq^{m/r}, t^r|q), \quad (m = 0, \dots, r-1), \quad (g^{(0)}(s, t^r|q) = g(s, t^r|q)). \quad (5.15)$$

$$g^{(m)}(s, t|q) = \mathbf{e}\left(\frac{mz}{r}\right) \phi\left(u + \frac{m\tau}{r}, z|\tau\right), \quad t = \mathbf{e}(z/r), \quad s = \mathbf{e}(u), \quad q = \tilde{q}^r.$$

They have the quasi-periodicities

$$g^{(m)}(s, t \cdot \omega|q) = \omega^m g^{(m)}(s, t|q), \quad (5.16)$$

$$s \tilde{q}^m g^{(m)}(s, t \tilde{q}|q) - g^{(m)}(s, t|q) = \delta(t, 1) - 1. \quad (5.17)$$

Similarly to (B.3) we have

$$\text{Res } g^{(m)}(s, t|q)|_{s=1} = 1, \quad \text{Res } g^{(m)}(s, t|q)|_{t=1} = 1. \quad (5.18)$$

In terms of these functions the other components of the Lax operators can be written as

$$L_m(t) = \sum_{k=1}^n S_m^k g^{(-m)}(1, t^r|q) \mathfrak{e}_k^m + \sum_{\tilde{\alpha} \in R_s} S_{\tilde{\alpha}}^{\tilde{\alpha}} g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u}), t^r|q) \mathfrak{t}_{\tilde{\alpha}}^m. \quad (5.19)$$

Due to (5.16) and (5.17) these components have the proper quasi-periodicities (5.11).

For the $E_{\tilde{G}}$ bundles we should replace \mathbf{u} on $\mathbf{u} + \varpi_j$, where ϖ_j are solutions of (4.69) taken from Table 2. Then from (5.13) and (5.19) we find

$$L_0(t) = \sum_{k=1}^n (v_k + S_0^k E_1(t^r|q)) \mathfrak{e}_k^0 + \sum_{\tilde{\alpha} \in \tilde{R}} S_{\tilde{\alpha}} g(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), t^r|q) E_{\tilde{\alpha}}, \quad (5.20)$$

$$L_m(t) = \sum_{k=1}^n S_m^k g^{(-m)}(1, t^r|q) \mathfrak{e}_k^m + \sum_{\tilde{\alpha} \in R_s} S_{\tilde{\alpha}}^{\tilde{\alpha}} g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), t^r|q) \mathfrak{t}_{\tilde{\alpha}}^m. \quad (5.21)$$

Remark 5.1 In the similar way we can consider meromorphic Higgs field on $C_{\tilde{q}}$ with $n > 1$ simple poles. In this way, instead of the twisted elliptic Calogero-Moser systems, that will be constructed below, we come to the twisted Gaudin systems. The quantum non-autonomous analog of the latter system is the Knizhnik-Zamolodchikov-Bernard equations. They will be considered in last section.

5.2 Hamiltonians

The quadratic Hamiltonian H can be found from the expansion of the $\text{tr} L^2(t)$. It is double-periodic function on $C_{\tilde{q}}$ with at most second order pole. Therefore,

$$\frac{1}{2} \text{tr} (L(t))^2 = H + E_2(t) C_2, \quad (5.22)$$

where C is the second Casimir function corresponding to the coadjoint G -orbit

$$C_2 = \frac{1}{2} \sum_{m=0}^{r-1} (\mathbf{S}, \mathbf{S}).$$

To calculate it we use (C.17), (C.26)

$$\frac{1}{2} \sum_{m=0}^{r-1} \text{tr} (L_m(t) L_{r-m}(t)) = \begin{cases} \frac{1}{2} (L_0(t), L_0(t)) + \frac{1}{2} (L_1(t) L_1(t)) & r = 2, \\ \frac{1}{2} (L_0(t) L_0(t)) + (L_1(t) L_2(t)) & r = 3. \end{cases}$$

Then it follows from (B.16) that

$$\begin{aligned} \frac{1}{2} \sum_{m=0}^{r-1} (L_m(t), L_{r-m}(t)) &= E_2(t) C_2(\mathbf{S}) + H(\mathbf{v}, \mathbf{u}, \mathbf{S}), \\ H(\mathbf{v}, \mathbf{u}, \mathbf{S}) &= H_0(\mathbf{v}, \mathbf{u}, \mathbf{S}) + H'(\mathbf{u}, \mathbf{S}), \end{aligned} \quad (5.23)$$

where

$$\begin{aligned} H_0(\mathbf{v}, \mathbf{u}, \mathbf{S}) &= \text{const. part } \frac{1}{2} (L_0(t), L_0(t)), \\ H'(\mathbf{u}, \mathbf{S}) &= \begin{cases} \text{const. part } \frac{1}{2} (L_1(t) L_1(t)) & r = 2, \\ \text{const. part } \frac{1}{2} (L_1(t) L_2(t)) & r = 3. \end{cases} \\ H_0(\mathbf{v}, \mathbf{u}, \mathbf{S}) &= \frac{1}{2} \sum_{k=1}^n v_k^2 + \sum_{\tilde{\alpha} \in \tilde{R}} S_0^{\tilde{\alpha}} S_0^{-\tilde{\alpha}} E_2(\mathbf{u}_{\tilde{\alpha}}), \\ H'(\mathbf{u}, \mathbf{S}) &= \sum_{m \neq 0} \left(-\frac{1}{2} \sum_{k=1}^n S_k^m S_k^{r-m} E_2(m\tau/r) + \sum_{\tilde{\alpha} \in R_s} S_m^{\tilde{\alpha}} S_{r-m}^{-\tilde{\alpha}} E_2(\mathbf{u}_{\tilde{\alpha}} - m\tau/r) \right). \end{aligned}$$

For the $E_{\tilde{G}}$ -bundles we have

$$\begin{aligned} H_0(\mathbf{v}, \mathbf{u}, \mathbf{S}) &= \frac{1}{2} \sum_{k=1}^n v_k^2 + \sum_{\tilde{\alpha} \in \tilde{R}} S_0^{\tilde{\alpha}} S_0^{-\tilde{\alpha}} E_2(\mathbf{u}_{\tilde{\alpha}} + \varpi_{j\tilde{\alpha}}), \\ H'(\mathbf{u}, \mathbf{S}) &= \sum_{m \neq 0} \left(-\frac{1}{2} \sum_{k=1}^n S_k^m S_k^{r-m} E_2(m\tau/r) + \sum_{\tilde{\alpha} \in R_s} S_m^{\tilde{\alpha}} S_{r-m}^{-\tilde{\alpha}} E_2(\mathbf{u}_{\tilde{\alpha}} + \varpi_{j\tilde{\alpha}} - m\tau/r) \right). \end{aligned}$$

5.3 CM system corresponding to $\mathcal{A}d_{SL}^{\nu, \Lambda}(C_q)$ bundle

Similarly to (5.9) the Lax operator is fixed by the conditions

$$1. \nu(L(t)) = L(-t), \quad 2. L(q^{-1}t) = Ad_{\mathbf{e}(\mathbf{u})\Lambda} L(t), \quad 3. Res_{t=1} L(t) = \mathbf{S}.$$

Here \mathbf{S} is decomposed in the basis (4.40)–(4.43)

$$\mathbf{S} = \sum_{a,b=0,1} \left(\sum_{j,k=1}^l \sum_{\alpha=0,1} S_{(a,b),jk}^{\alpha} \mathcal{T}_{(ab),jk}^{\alpha} + \sum_{j=1}^l S_{(a,b),j} \mathcal{H}_{(a,b),j} \right). \quad (5.24)$$

According with (4.54) decompose $L(t)$ as

$$L(t) = \sum_{\alpha,a,b=0,1} L_{(a,b)}^{\alpha}(t) + \sum_{a,b=0,1} L_{(a,b)}^{diag}(t).$$

Consider the diagonal part of the Lax operator. If $a = b = 0$ then

$$L_{(0,0)}^{diag}(t) = \sum_{j=1}^l (v_j + S_{(0,0),j} E_1(t^2|q)) \mathcal{H}_{(0,0),j}.$$

Otherwise,

$$L_{(a,b)}^{diag}(t) = \sum_{j=1}^l S S_{(0,0),j} t^b g(\tilde{q}^b \mathbf{e}(a/2), t^2|q) \mathcal{H}_{(a,b),j}.$$

The non-diagonal part is

$$L_{(a,b)}^{\alpha}(t) = \sum_{j,k=1}^l L_{(a,b),jk}^{\alpha}(t) \mathcal{T}_{(ab),jk}^{\alpha},$$

where the component satisfy the quasi-periodicity conditions

$$L_{(a,b),jk}^{\alpha}(-t) = (-1)^b L_{(a,b),jk}^{\alpha}(t), \quad L_{(a,b),jk}^{\alpha}(\tilde{q}t) = (-1)^a \mathbf{e}(u_{jk}) L_{(a,b),jk}^{\alpha}(t).$$

Then we find

$$L_{(a,b),jk}(t) = S_{(a,b),jk}^{\alpha} t^b g(\mathbf{e}(-u_{jk} + a/2) \tilde{q}^b, t^2|q^2) = S_{(a,b),jk}^{\alpha} \mathbf{e}(bz/2) \phi(-u_{jk} + \frac{a+b\tau}{2}, z|\tau).$$

The Hamiltonian we find from the expansion (5.22) using the Killing form (4.46) and the Fay identity (B.16). As above, we assume that $S_{(0,0),j} = 0$ for $j = 1, \dots, l$. Then the Hamiltonian $H = \frac{1}{8}(\text{const. part tr } L^2(t))$ takes the form

$$\begin{aligned} H = & \frac{1}{2} \sum_{j=1}^l v_j^2 + \sum_{a+b>0} \sum_{j=1}^l S_{(0,0),j}^2 E_2\left(\frac{a+b\tau}{2}|\tau\right) + \sum_{\alpha,a,b} \left(\sum_{j \neq k, j \neq n-k+1}^l (S_{(a,b),jk}^{\alpha})^2 E_2(u_j \pm u_k + \frac{a+b\tau}{2}|\tau) \right. \\ & \left. + \sum_{j=1}^l (S_{(a,b),j}^{\alpha})^2 E_2(u_j + \frac{a+b\tau}{2}|\tau) + \sum_{j=1}^l (S_{(a,b),j}^{\alpha})^2 E_2(2u_j + \frac{a+b\tau}{2}|\tau) \right). \end{aligned} \quad (5.25)$$

6 Classical r-matrices and KZB equations

6.1 r-matrix

The classical r -matrix is the map $C_q \times C_q \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, satisfying the classical dynamical Yang-Baxter equation (CDYBE) (see below). More exactly, it is a section of $\mathfrak{g} \otimes \mathfrak{g}$ bundle over the moduli space $Bun_{\sigma, \zeta}(Ad_G(C_q))$ (4.32), or $Bun_{\sigma, \zeta}(E_{\bar{G}}(C_{\bar{q}}))$ (4.67) defined by the properties of the coefficients $g^{(-m)}(\frac{m}{r}, t^r | q)$ (5.16) and (5.18).

It can be defined by means of the Lax operator as

$$r(t) = \frac{\partial L_1(t)}{\partial S_2}, \quad (6.1)$$

Starting with the Lax operator corresponding to the non-trivial characteristic classes (5.20), (5.21) we come to the r -matrix

$$r(\mathbf{u}, t) = \sum_{m=0}^{r-1} r_m(t), \quad (r = 2, 3), \quad r_m(t\omega, s) = \omega^{-m} r_m(t). \quad (6.2)$$

$$r_0(\mathbf{u}, t) = \sum_{k=1}^n E_1(t^r | q) \mathfrak{e}_k^0 \otimes \mathfrak{e}_k^0 + \sum_{\tilde{\alpha} \in \tilde{R}} |\tilde{\alpha}|^2 g(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), t^r | q) E_{\tilde{\alpha}} \otimes E_{\tilde{\alpha}}, \quad (6.3)$$

$$r_m(\mathbf{u}, t) = \sum_{k=1}^n g^{(-m)}(1, t^r | q) (\mathfrak{e}_k^m \otimes \mathfrak{e}_k^{r-m}) + \frac{1}{r} \sum_{\tilde{\alpha} \in R_s} |\tilde{\alpha}|^2 g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), t^r | q) (\mathfrak{t}_{\tilde{\alpha}}^m \otimes \mathfrak{t}_{-\tilde{\alpha}}^{r-m}), \quad (6.4)$$

where $E_{\tilde{\alpha}}$ is defined by (C.13), $g^{(-m)}(\frac{m}{r}, t^r | q)$ is defined by (5.15).

As in [28] it can be proved that $r(t)$ satisfies classical dynamical Yang-Baxter equation:

$$[r_{12}(t_1, t_2), r_{13}(t_1, t_3)] + [r_{12}(t_1, t_2), r_{23}(t_2, t_3)] + [r_{13}(t_1, t_3), r_{23}(t_2, t_3)] - \quad (6.5)$$

$$\sqrt{r} \sum_{m=0}^{r-1} \sum_{\tilde{\alpha} \in R_s} \frac{|\tilde{\alpha}|^2}{2} \left(\mathfrak{t}_{\tilde{\alpha}}^m \otimes \mathfrak{t}_{-\tilde{\alpha}}^{r-m} \otimes \mathfrak{e}_{\tilde{\alpha}}^0 \partial_{\tilde{\alpha}(\mathbf{u})} g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), (t_1/t_2)^r | q) - \right.$$

$$\left. \mathfrak{t}_{\tilde{\alpha}}^m \otimes \mathfrak{e}_{\tilde{\alpha}}^0 \otimes \mathfrak{t}_{-\tilde{\alpha}}^{r-m} \partial_{\mathbf{u}} g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), t_1^r/t_3^r | q) + \mathfrak{e}_{\tilde{\alpha}}^0 \otimes \mathfrak{t}_{\tilde{\alpha}}^m \otimes \mathfrak{t}_{-\tilde{\alpha}}^{r-m} \partial_{\mathbf{u}} g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), t_2^r/t_3^r | q) \right) = 0.$$

It follows from (B.7) and (B.12) that the r -matrix plays the role of the Green function $G(t/t_1)$ of the equation (4.23).

The r -matrix (6.2)-(6.4) and described above the Lax operator (5.20), (5.21) define the Poisson brackets (5.7) via RLL -equation:

$$\{L(t_1) \otimes 1, 1 \otimes L(t_2)\} = [L(t_1) \otimes 1 + 1 \otimes L(t_2), r(t_1/t_2)] - \quad (6.6)$$

$$- \frac{\sqrt{r}}{2} \sum_{m=0}^{r-1} \sum_{\tilde{\alpha} \in R_s} |\tilde{\alpha}|^2 \partial_{\mathbf{u}} g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u}), (t_1/t_2)^r | q) (S^h)_0^{\tilde{\alpha}} \mathfrak{t}_{\tilde{\alpha}}^m \otimes \mathfrak{t}_{-\tilde{\alpha}}^{r-m}.$$

The anomalous last term vanishes upon the moment constraints $(S^h)_0^{\tilde{\alpha}} = 0$.

6.2 KZB equations

Let \mathfrak{M}_l be the space of complex structure of the elliptic curve with l marked points (t_1, \dots, t_l) . An open cell in \mathfrak{M}_l is described by the coordinates $(q = \mathbf{e}(\tau), t_1, \dots, t_l, \prod t_j = 1)$. The KZB equations define a flat connection in the bundle of conformal blocks in the WZW theory on C_q over \mathfrak{M}_l [13, 14, 15]. In our case the variables in the WZW theory are defined by the quasi-periodic conditions (4.2), (4.3). Acting as in [29] we come to the following description of the KZB equations.

Consider the following differential operators:

$$\nabla_a = t_a \partial_{t_a} + \hat{\partial}^a + \sum_{c \neq a} r^{ac}, \quad (6.7)$$

$$\nabla_\tau = 2\pi i \partial_\tau + \Delta + \frac{1}{2} \sum_{b,d} f^{bd}, \quad (6.8)$$

with

$$\begin{aligned} r^{ac} &= \sum_{m=1}^{r-1} \sum_{\tilde{\alpha} \in \tilde{R}} |\alpha|^2 g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), (t_a/t_c)^r) \mathfrak{t}_{\tilde{\alpha}}^{m,a} \otimes \mathfrak{t}_{-\tilde{\alpha}}^{-m,c} + \sum_{m=0}^{r-1} \sum_{k=1}^n g^{(-m)}(1, (t_a/t_c)^r) \mathfrak{e}_k^{m,a} \otimes \mathfrak{e}_k^{-m,c}, \\ f^{ac} &= \sum_{m=1}^{r-1} \sum_{\tilde{\alpha} \in \tilde{R}} |\alpha|^2 \partial_{\tilde{\alpha}(\mathbf{u})} g^{(-m)}(s_{\tilde{\alpha}}^{-1}(\mathbf{u} + \varpi_j), (t_a/t_c)^r) \mathfrak{t}_{\tilde{\alpha}}^{m,a} \otimes \mathfrak{t}_{-\tilde{\alpha}}^{-m,c} + \\ &\quad \sum_{k=1}^n \sum_{m=1}^{r-1} g^{(-m)}(1, (t_a/t_c)^r) \mathfrak{e}_k^{m,a} \otimes \mathfrak{e}_k^{-m,c} + \frac{1}{2} (E_1^2(t_a^r/t_c^r) - \wp(t_a^r/t_c^r)) \mathfrak{e}_k^{0,a} \otimes \mathfrak{e}_k^{0,c}, \end{aligned}$$

where $\mathfrak{t}_\alpha^{k,a} = 1 \otimes \dots \otimes \mathfrak{t}_\alpha^k \otimes 1 \dots \otimes 1$ (with \mathfrak{t}_α^k on the a -th place) and similarly for the generators $\mathfrak{e}_k^{m,a}$ ² and the Weierstrass function \wp is defined in (B.15). The following short notations are used here:

$$\hat{\partial}^a = r \sum_{\tilde{\alpha} \in \tilde{\Pi}} \mathfrak{e}_{\tilde{\alpha}} \partial_{\tilde{\alpha}}, \quad \Delta = \frac{r}{2} \sum_{\tilde{\alpha} \in \tilde{\Pi}} \partial_{u(\tilde{\alpha})}^2, \quad g^{(0)}(1, (t_a/t_c)^r) = E_1((t_a/t_c)^r).$$

As in [29] it can be proved that the connection (6.7), (6.8) is flat.

7 Appendix A: Structure of Table 2

The group $\mathcal{Z}(\bar{G}) = P/Q$ can be defined in terms of the Cartan matrix $a_{jk} = (\alpha_j, \alpha_k)$. Since the fundamental weights ϖ_j , generating the weight lattice P , form a basis in \mathfrak{h}^* dual to the basis of simple roots $(\alpha_j, \varpi_k) = \delta_{jk}$, we have the following relations between two bases in the lattices P and Q $\varpi_k = (a)_{kj}^{-1} \alpha_j$. In particular, $\text{ord}(\mathcal{Z}(\bar{G})) = \det a$.

It follows from (4.17) that the fundamental weights ϖ_j form the basis

$$\{\eta_j = \sigma(\varpi_j) - \varpi_j\}, \quad (A.1)$$

in P^\vee , while elements of the root lattice $Q(\mathfrak{g})$ of the form

$$\{\delta_j = \sigma(\alpha_j) - \alpha_j \mid \alpha_j \in \Pi(\mathfrak{g})\} \quad (A.2)$$

²For brevity we write $\mathfrak{t}_\alpha^{k,a}$, $\mathfrak{e}_k^{m,a}$ instead of representations of these generators in the spaces V_{μ_a} .

form a bases in Q^Υ . By comparison of these two bases we find the structure of quotient group $\mathbb{Z}^\sigma(\bar{G})$.

We construct Table 2 line by line.

1) $\text{SL}(N, \mathbb{C})$,

We use the canonical basis (e_1, \dots, e_N) . Since the fundamental weights of A_{N-1} are

$$\begin{aligned}\varpi_1 &= \left(\frac{N-1}{N}, -\frac{1}{N}, \dots, -\frac{1}{N}\right), \\ \varpi_2 &= \left(\frac{N-2}{N}, \frac{N-2}{N}, \dots, -\frac{2}{N}\right), \\ &\dots\dots \\ \varpi_j &= \left(\frac{N-j}{N}, \dots, \frac{N-j}{N}, -\frac{j}{N}, \dots, -\frac{j}{N}\right), \\ &\dots\dots \\ \varpi_{N-1} &= \left(\frac{1}{N}, \frac{1}{N}, \dots, \frac{1-N}{N}\right).\end{aligned}$$

we find generators of $P^\Upsilon(A_{N-1})$

$$\eta_j = - \left(\underbrace{1 - \frac{2j}{N}, \dots, 1 - \frac{2j}{N}}_j, \underbrace{-\frac{2j}{N}, \dots, -\frac{2j}{N}}_{N-2j}, \underbrace{1 - \frac{2j}{N}, \dots, 1 - \frac{2j}{N}}_j \right), \quad (\text{A.3})$$

where

$$j = \begin{cases} 1, \dots, n-1 & \text{for } N = 2n, \\ 1, \dots, n & \text{for } N = 2n+1. \end{cases}$$

On the other hand, since $\Pi(A_{N-1}) = \{e_j - e_{j+1} \mid j = 1, \dots, N-1\}$ we obtain from (A.2)

$$\delta_j = \begin{cases} (0, \dots, -1, 1, 0, \dots, 0, 1, -1, 0, \dots, 0) & j = 1, \dots, n-1, \quad N = 2n, 2n+1, \\ (0, \dots, -1, 2, -1, \dots, 0) & j = n, \quad N = 2n+1. \end{cases} \quad (\text{A.4})$$

It follows from (A.3) and (A.4) the quotient group $\mathbb{Z}^\sigma(\text{SL}(N, \mathbb{C})) = P^\Upsilon(A_{N-1})/Q^\Upsilon(A_{N-1})$ is generated by the element η_1 . For $N = 2n$ it has the order n , while for the odd N the quotient has the order N . In this way we have derived the first two lines in the Table.

2. $\text{SO}(2n+2, \mathbb{C}), (D_{n+1})$

In the Cartan subalgebra $\mathfrak{h}(D_{n+1})$ we choose a canonical bases (e_1, \dots, e_{n+1}) and the system of simple roots

$$\Pi(D_{n+1}) = \{\alpha_j = e_j - e_{j+1}, \quad j = 1, \dots, n, \quad \alpha_{n+1} = e_n + e_{n+1}\}. \quad (\text{A.5})$$

The dual system of the fundamental weights takes the form

$$\varpi_j = e_1 + e_2 + \dots + e_j, \quad j = 1, \dots, n-1, \quad (\text{A.6})$$

$$\varpi_n = \frac{1}{2}(e_1 + \dots + e_n - e_{n+1}), \quad \varpi_{n+1} = \frac{1}{2}(e_1 + \dots + e_n + e_{n+1}).$$

The outer automorphism acts as

$$\sigma(\alpha_j) = \begin{cases} \alpha_j & 1 \leq j < n, \\ \alpha_n \leftrightarrow \alpha_{n+1}, \end{cases} \quad \sigma(\varpi_j) = \begin{cases} \varpi_j & 1 \leq j < n, \\ \varpi_n \leftrightarrow \varpi_{n+1}. \end{cases}$$

Following (A.1), (A.2) we find generators of $P^\Upsilon(D_{n=1})$ and $Q^\Upsilon(D_{n=1})$

$$\eta_n = \sigma(\varpi_n) - \varpi_n = e_{n+1}, \quad \delta = 2e_{n+1}. \quad (\text{A.7})$$

Thus, the quotient group $\mathbb{Z}^\sigma(\text{Spin}_{2n+2}) \sim \mathbb{Z}_2$ is generated by η (A.7).

3. $\text{Spin}_8, (D_4)$

Here we have the outer automorphism of order 3. In terms of notations (A.5) and (A.6) for $n = 3$ it acts as

$$\begin{aligned}\sigma(\alpha_1) &= \alpha_3, \quad \sigma(\alpha_2) = \alpha_2, \quad \sigma(\alpha_3) = \alpha_4, \quad \sigma(\alpha_4) = \alpha_1, \\ \sigma(\varpi_1) &= \varpi_3, \quad \sigma(\varpi_2) = \varpi_2, \quad \sigma(\varpi_3) = \varpi_4, \quad \sigma(\varpi_4) = \varpi_1,\end{aligned}$$

Then P^Υ is generated by the weights

$$\eta_1 = \varpi_1 - \varpi_3 = \frac{1}{2}(1, -1, -1, 1), \quad \eta_2 = \varpi_3 - \varpi_4 = -(0, 0, 0, 1), \quad (\text{A.8})$$

while Q^Υ is generated by the elements

$$\delta_1 = (1, -1, -1, 1), \quad \delta_2 = -2(0, 0, 0, 1)$$

Then η_1 and η_2 are generators of the two groups $\mathbb{Z}^\sigma(\text{Spin}_4) = P^\Upsilon(D_4)/Q^\Upsilon(D_4) \sim \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

4. E_6

The outer automorphism of the Lie algebra E_6 corresponds to the symmetry of the simple roots $\Pi(E_6) = (\alpha_j)$

$$\sigma : \alpha_1 \leftrightarrow \alpha_5, \quad \alpha_2 \leftrightarrow \alpha_4, \quad \alpha_{3,6} \leftrightarrow \alpha_{3,6},$$

The symmetry of the fundamental weights

$$\begin{aligned}\varpi_1 &= \frac{1}{3}(4\alpha_1 + 5\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6), \\ \varpi_2 &= \frac{1}{3}(5\alpha_1 + 10\alpha_2 + 12\alpha_3 + 8\alpha_4 + 4\alpha_5 + 6\alpha_6), \\ \varpi_3 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6, \\ \varpi_4 &= \frac{1}{3}(4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 10\alpha_4 + 5\alpha_5 + 6\alpha_6), \\ \varpi_5 &= \frac{1}{3}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6), \\ \varpi_6 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6.\end{aligned}$$

has the similar form

$$\sigma : \varpi_1 \leftrightarrow \varpi_5, \quad \varpi_2 \leftrightarrow \varpi_4, \quad \varpi_{3,6} \leftrightarrow \varpi_{3,6}.$$

The lattice $P^\Upsilon(E_6)$ is generated by the two elements of weight lattice $P(E_6)$

$$\eta_1 = \varpi_1 - \varpi_5 = \frac{1}{3}(2\alpha_1 + \alpha_2 - \alpha_4 - 2\alpha_5), \quad \eta_2 = \varpi_2 - \varpi_4 = \frac{1}{3}(\alpha_1 + 2\alpha_2 - 2\alpha_4 - \alpha_5). \quad (\text{A.9})$$

and $Q^\Upsilon(E_6)$ by the two elements of the root lattice $Q(E_6)$

$$\delta_1 = \alpha_1 - \alpha_5, \quad \delta_2 = \alpha_2 - \alpha_4.$$

The quotient group $\mathbb{Z}^\sigma(E_6) = P^\Upsilon(E_6)/Q^\Upsilon(E_6) \sim \mathbb{Z}_3$ is generated by one of the elements $\eta_{1,2}$ (A.9).

8 Appendix B: Some elliptic functions [2, 36]

The main ingredient is the Kronecker function on $s, t \in \mathbb{C}^*$

$$g(s, t|q) = - \sum_{m \in \mathbb{Z}} \frac{t^m}{1 - q^m s}, \quad \text{for } 1 > |t| > |q|. \quad (\text{B.1})$$

If in addition $1 > |s| > |q|$ then

$$g(s, t|q) = 1 - \frac{1}{1-t} - \frac{1}{1-s} + \sum_{i,n < 0} s^i q^{in} t^n - \sum_{i,n > 0} s^i q^{in} t^n. \quad (\text{B.2})$$

Then

$$\text{Res } g(s, t|\tilde{q})|_{s=1} = 1, \quad \text{Res } g(s, t|q)|_{t=1} = 1. \quad (\text{B.3})$$

Rewrite (B.2) as

$$g(s, t|q) = 1 + g^+(t) + g^-(t), \quad (\text{for } 1 > |s| > |q|, \quad 1 > |t| > |q|), \quad (\text{B.4})$$

$$g^+(s, t) = - \sum_{i,n \geq 0} s^i q^{in} t^n, \quad g^-(s, t) = \sum_{i,n < 0} s^i q^{in} t^n.$$

Consider the distribution on the space of the Laurent polynomials $\mathbb{C}[t, t^{-1}] = \{\psi(t) = \sum_l c_l t^l\}$, defined by the functional $\delta(t, t_1)$

$$\text{Res}_{t_1=0} (\delta(t, t_1) \psi(t_1)) = \psi(t). \quad (\text{B.5})$$

It is represented as formal power series

$$\begin{aligned} \delta(t_1, t_2) &= \sum_{k \in \mathbb{Z}} t_1^k t_2^{-k-1} = \delta_+(t_1, t_2) + \delta_-(t_1, t_2), \\ \delta_+(t_1, t_2) &= \frac{1}{t_2} \sum_{n \geq 0} (t_1/t_2)^n = \frac{1}{t_1 - t_2} \text{ for } |t_2| < |t_1|, \\ \delta_-(t_1, t_2) &= \frac{1}{t_2} \sum_{n > 0} (t_2/t_1)^n = -\frac{1}{t_1 - t_2} \text{ for } |t_2| > |t_1|. \end{aligned}$$

Fix $t \in \mathbb{C}^*$. Define two contours in \mathbb{C}^* with the opposite orientation:

$$\gamma_1 = \{|s| = |t|(1 + \varepsilon)\}, \quad \gamma_2 = \{|s| = |t|(1 - \varepsilon)\}, \quad \varepsilon > 0,$$

and let $\gamma = \gamma_1 \cup \gamma_2$. Then in accordance with (B.5)

$$\psi(t) = \frac{1}{2\pi i} \oint_{\gamma} \psi(t_1) \delta(t, t_1) \frac{dt_1}{t_1}. \quad (\text{B.6})$$

It follows from (B.1) that the Kronecker function plays the role of the Green function for the difference operator

$$sg(s, qt|q) - g(s, t|q) = \delta(t, 1) - 1. \quad (\text{B.7})$$

Let $\psi(t)$ be holomorphic function in a neighborhood of the contour γ , such that $\text{Res}_{t=1} \frac{\psi(t)}{t} = 0$. Then there exists a holomorphic in \mathbb{C}^* solution $x(t)$ of the difference equation

$$sx(qt) - x(t) = \psi(t). \quad (\text{B.8})$$

It is defined by the integral

$$x(t) = \frac{1}{2\pi i} \oint_{\gamma_1} \psi(t_1) g(s, t/t_1|q) dt_1/t_1. \quad (\text{B.9})$$

In terms of the theta function the Kronecker function can be represented as

$$g(s, t|q) = \frac{X(ts|q)P^2(q)}{X(t|q)X(s|q)}, \quad P(q) = \prod_{n>0} (1 - q^n),$$

where

$$X(t|q) = 4\iota q^{1/8} (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \prod_{n \geq 1} (1 - q^n t)(1 - q^n t^{-1})(1 - q^n) = \iota \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} t^{(n-1)/2}.$$

In the variables (u, z) , $(s = \mathbf{e}(u), t = \mathbf{e}(z))$ $X(t|q)$ is the odd theta-function $X(t|q)|_{t=\mathbf{e}(z)} = \theta(z|\tau)$, and

$$g(s, t|q)|_{s=\mathbf{e}(u), t=\mathbf{e}(z)} = \phi(u, z|\tau) = \frac{\theta(u+z)\theta'(0)}{\theta(u)\theta(z)}.$$

The first Eisenstein series are defined as

$$E_1(z|\tau) = \partial_z \log \theta(z|\tau).$$

It has a simple poles at the lattice

$$\text{Res } E_1(z|\tau)|_{z=m+n\tau} = 1. \quad (\text{B.10})$$

In the variable $t = \mathbf{e}(z)$ it can be defined on the annulus $1 > |t| > |q|$ as the series

$$E_1(t|q) = \pi\iota + 2\pi\iota \sum_{m \neq 0} \frac{t^m}{q^m - 1}. \quad (\text{B.11})$$

The function $-\frac{1}{2\pi\iota} E_1(t|q)$ plays the role of the Green function for the difference operator on C_q

$$-\frac{1}{2\pi\iota} E_1(tq|q) + \frac{1}{2\pi\iota} E_1(t|q) = \delta(t, 1) - 1, \quad (\text{B.12})$$

where $\delta(t, 1)$ is (B.6). It allows one to continue $E_1(t|q)$ from C_q on \mathbb{C}^* .

Let $E_2(t|q)$ be the second Eisenstein series:

$$E_2(t|q) = 4\pi^2 \sum_n \frac{q^n t}{(1 - q^n t)^2} = -\partial_z^2 \log \theta(z|q). \quad (\text{B.13})$$

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log \vartheta(z|\tau). \quad (\text{B.14})$$

Let $\eta(q) = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n)$, and

$$\eta_1(q) = \frac{24}{2\pi\iota} \frac{\partial_\tau \eta(q)}{\eta(q)}.$$

Then the Weierstrass function is defined as

$$\wp(q, \tau) = E_2(q, \tau) - 2\eta_1(q). \quad (\text{B.15})$$

The function g satisfies the Fay identities:

$$g(s, t|q)g(s^{-1}, t|q) = E_2\left(\frac{1}{2\pi\iota} \ln t|q\right) - E_2\left(\frac{1}{2\pi\iota} \ln s|q\right), \quad (\text{B.16})$$

$$\begin{aligned} & g(s_1, t|q)s_2 \partial_{s_2} g(s_2, t|q) - g(s_2, t|q)s_1 \partial_{s_1} g(s_1, t|q) = \\ & \frac{1}{2\pi\iota} \left(E_2\left(\frac{1}{2\pi\iota} \ln s_2|q\right) - E_2\left(\frac{1}{2\pi\iota} \ln s_1|q\right) \right) g(s_1 s_2, t|q). \end{aligned} \quad (\text{B.17})$$

9 Appendix C: Outer automorphisms of simple algebras [22, 23, 35]

Let R be the root system of the Lie algebra \mathfrak{g} of rank l , $W(R)$ is the Weyl group and $Aut(R) \supseteq W(R)$ is the group of the automorphisms of R . The group $Out(\mathfrak{g}) = \{\nu\}$ is isomorphic to the quotient group $Aut(\Pi) \sim Aut(R)/W(R)$ (see (3.1)). This group is non-trivial for the simply-laced algebras

$$\begin{aligned} R = A_l, D_l, E_6 & \quad Aut(R)/W(R) \sim \mathbb{Z}_2, \\ R = D_4 & \quad Aut(R)/W(R) \sim \mathbb{Z}_3. \end{aligned} \quad (C.1)$$

The outer automorphisms provide the decompositions of \mathfrak{g} on its eigenspaces

$$\mathfrak{g} = \bigoplus_{m=0}^{r-1} \mathfrak{g}_m, \quad \nu(\mathfrak{g}_m) = \omega^m \mathfrak{g}_m, \quad (\omega = \exp(\frac{2\pi i}{r})), \quad r = 2, \quad (\text{or } r = 3 \text{ for } D_4). \quad (C.2)$$

The subspaces \mathfrak{g}_0 is an invariant subalgebra of \mathfrak{g} and \mathfrak{g}_j are irreducible modules of \mathfrak{g}_0 .

Let \mathfrak{h} be the Cartan subalgebra, E_α are the root subspaces and

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathbb{C}E_\alpha$$

is the root decomposition of \mathfrak{g} . The action of the group $Aut(\mathfrak{g}, \mathfrak{h}, \Pi)$ (3.1) preserves this decomposition. Then we define the action of ν on R as $\nu \rightarrow \bar{\nu}\alpha$, $\bar{\nu} \in Aut(\Pi)$

$$\nu E_\alpha = E_{\bar{\nu}\alpha}, \quad (\alpha \in R = R^\vee).^3 \quad (C.3)$$

Represent the Cartan subalgebra as $\mathfrak{h} = \mathfrak{h}^{\mathbb{R}} + i\mathfrak{h}^{\mathbb{R}}$ and let

$$\Pi = (\alpha_1, \dots, \alpha_l), \quad (l = \text{rank } \mathfrak{g}) \quad (C.4)$$

be a system of simple roots generating R . Define the Weyl chamber in $\mathfrak{h}^{\mathbb{R}}$

$$C^+ = \{x \in \mathfrak{h}^{\mathbb{R}} \mid \langle x, \alpha \rangle > 0 \text{ for } \alpha \in \Pi\}. \quad (C.5)$$

Define the invariant basis in the homogeneous component \mathfrak{g}_m of \mathfrak{g} . Under the action of $\bar{\nu}$ the set of roots R is splitted on the orbits on the lengths 1 (invariant subset), O_2 of the length 2 and O_3 of the length 3 (for D_4). For all \mathfrak{g} with non-trivial outer automorphisms except $\mathfrak{sl}(2n+1)$ (A_{2n}) define two subsets of roots

$$\begin{aligned} R_l &= \{\tilde{\alpha} = \alpha \in R \mid \bar{\nu}(\alpha) = \alpha\}, \\ R_s &= \{\tilde{\alpha} = \frac{1}{r} \sum_{m=0}^{r-1} \bar{\nu}^m(\alpha) \mid \alpha \in R, \bar{\nu}(\alpha) \neq \alpha\}. \end{aligned} \quad (C.6)$$

The subset $\tilde{R} = R_l \cup R_s$ is the set of roots of the invariant subalgebra \mathfrak{g}_0 with respect to the Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$. The system of simple roots

$$\tilde{\Pi} = \{\tilde{\alpha}_j, \quad j = 1, \dots, n\} \quad (C.7)$$

of the invariant algebra \mathfrak{g}_0 is the basis of \tilde{R} .

Let $\tilde{\alpha}_0$ be the highest weight of the \mathfrak{g}_1 module. It is decomposed in the root basis as

$$\tilde{\alpha}_0 = \sum_{j=1}^n a_j \tilde{\alpha}_j, \quad (C.8)$$

³Since the algebras are simply-laced their root and weight systems coincide with the coroots and coweight systems. The out-invariant subalgebras are not simply-laced and these systems are different.

where n is a rank of \mathfrak{g}_0 . The extended root system

$$\tilde{\Pi}^{ext} = \tilde{\Pi} \cup (-\tilde{\alpha}_0) \quad (\text{C.9})$$

defines twisted affine Dynkin diagram.

Let

$$\mathfrak{t}_{\tilde{\alpha}}^m = \sum_{j=0}^{r-1} \omega^{jm} E_{\nu^j \alpha}, \quad (\tilde{\alpha} \in R_s), \quad \nu(\mathfrak{t}_{\tilde{\alpha}}^m) = \omega^m \mathfrak{t}_{\tilde{\alpha}}^m. \quad (\text{C.10})$$

$$\mathfrak{h} = \sum_{m=0}^{r-1} \mathfrak{h}_m, \quad \nu(\mathfrak{h}_m) = \omega^m \mathfrak{h}_m. \quad (\text{C.11})$$

Then the subspace \mathfrak{g}_m in (C.2) has the root decomposition

$$\mathfrak{g}_m = \mathfrak{h}_m + \sum_{\tilde{\alpha} \in R_s} \mathbb{C} \mathfrak{t}_{\tilde{\alpha}}^m, \quad m \neq 0. \quad (\text{C.12})$$

$$\mathfrak{g}_0 = \mathfrak{h}_0 + \sum_{\tilde{\alpha} \in \tilde{R}} \mathbb{C} E_{\tilde{\alpha}}, \quad E_{\tilde{\alpha}} = \begin{cases} E_{\alpha} & \alpha = \tilde{\alpha} \in R_l, \\ t_{\tilde{\alpha}}^0 & \tilde{\alpha} \in R_s. \end{cases} \quad (\text{C.13})$$

In these terms the set of the root subspaces of \mathfrak{g} is

$$E_{\tilde{\alpha}}, \quad (\tilde{\alpha} \in R_l), \quad t_{\tilde{\alpha}}^0 \quad (\tilde{\alpha} \in R_s), \quad \mathfrak{t}_{\tilde{\alpha}}^m, \quad (\tilde{\alpha} \in R_s).$$

The invariant subalgebra \mathfrak{g}_0 has the rank

$$\text{rank } \mathfrak{g}_0 = n, \quad \text{for } A_{2n}, A_{2n-1}, D_{n+1}$$

$$\text{rank } \mathfrak{g}_0 = 4, \quad \text{for } E_6, \quad \text{rank } \mathfrak{g}_0 = 2, \quad \text{for } D_4.$$

Let (e_1, \dots, e_l) be a canonical basis in \mathfrak{h} . The basis in \mathfrak{h}_m for $m \neq 0$ assumes the form $\mathfrak{h}_m = \langle \mathfrak{e}_1^m, \dots, \mathfrak{e}_n^m \rangle$,

$$\mathfrak{e}_k^m = \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} \omega^{jm} e_{\nu^j k} \quad \text{for } \nu(e_k) \neq e_k, \quad \nu(\mathfrak{e}_k^m) = \omega^m \mathfrak{e}_k^m, \quad (k = 1, \dots, n), \quad (\text{C.14})$$

where n is equal to the number of orbits of the ν action on \mathfrak{h} . In particular, the basis in \mathfrak{h}_0 is

$$\mathfrak{e}_k^0 = \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} e_{\nu^j k} \quad \text{for } \nu(e_k) \neq e_k, \quad \text{and } \mathfrak{e}_i^0 = e_i \quad \text{for } \nu(e_i) = e_i. \quad (\text{C.15})$$

Summarizing, the following set of the generators T_a ($a = (\tilde{\alpha}, m, k)$) defines the basis in \mathfrak{g}

$$T_a = \left\{ \begin{array}{ll} \mathfrak{g}_0 & \mathfrak{e}_k^0 (C.15), \quad E_{\tilde{\alpha}} (C.13), \\ \mathfrak{g}_m, \quad (m \neq 0) & \mathfrak{e}_k^m (C.14), \quad \mathfrak{t}_{\tilde{\alpha}}^m (C.10), \end{array} \quad m = 0, \dots, r-1, \quad k = 1, \dots, n \right\}. \quad (\text{C.16})$$

The Killing form on these generators takes the form

$$\begin{aligned} (\mathfrak{t}_{\tilde{\alpha}}^m, \mathfrak{t}_{\tilde{\beta}}^n) &= r \delta^{m+n, r} \delta_{\alpha, -\beta}, \quad m = 0, \dots, r-1, \\ (E_{\tilde{\alpha}}, E_{\tilde{\beta}}) &= \delta_{\tilde{\alpha}, -\tilde{\beta}}, \quad \tilde{\alpha} \in R_l, \\ (\mathfrak{e}_k^m, \mathfrak{e}_j^n) &= \delta^{m+n, r} \delta_{j, k}. \end{aligned} \quad (\text{C.17})$$

For $\mathfrak{g} = \mathfrak{sl}(2n+1)$ (A_{2n}) the invariant subalgebra $\mathfrak{g}_0 = B_n$.

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{g}_0 = \mathfrak{so}(2n+1),$$

The outer automorphism in the fundamental representation of the algebra $\mathfrak{sl}(2n+1)$ can be defined as

$$\nu(x) = -Jx^T J, \quad J = \begin{pmatrix} 0 & 0 & j \\ 0 & 1 & 0 \\ -j & 0 & 0 \end{pmatrix}, \quad j = (E_{k, n-k+1}). \quad (\text{C.18})$$

The anti-invariant part has the form

$$\mathfrak{g}_1 = \begin{pmatrix} A & \tilde{X}^T & B \\ Y & 0 & \tilde{X} \\ C & \tilde{Y}^T & \tilde{A}^T \end{pmatrix}, \quad \tilde{A} = jA j, \quad \tilde{X} = X j, \quad \tilde{Y} = Y j, \quad B = -\tilde{B}^T, \quad C = -\tilde{C}^T.$$

The root system of \mathfrak{g}_0 is $\tilde{R}_{B_n} = R_l \cup R_s$, where the short and long roots are

$$\begin{aligned} R_l &= \{ \tilde{\alpha} = \frac{1}{2}(\alpha + \nu(\alpha)) \mid \alpha \neq \nu(\alpha) \text{ and } (\alpha, \nu(\alpha)) = 0 \}, \\ R_s &= \{ \tilde{\alpha} = \frac{1}{2}(\alpha + \nu(\alpha)) \mid \alpha \neq \nu(\alpha) \text{ and } (\alpha, \nu(\alpha)) \neq 0 \}. \end{aligned} \quad (\text{C.19})$$

Let $ht(\alpha)$ be the height of $\alpha \in R$. The root spaces of \mathfrak{g}_0 are

$$E_{\tilde{\alpha}} = \begin{cases} E_{\alpha} - (-1)^{ht(\alpha)} E_{\nu(\alpha)} & \alpha \in R_l, \\ \sqrt{2}(E_{\alpha} - (-1)^{ht(\alpha)} E_{\nu(\alpha)}) & \alpha \in R_s. \end{cases} \quad (\text{C.20})$$

The other root subspaces are

$$\mathfrak{t}_{\tilde{\alpha}}^1 = \sum_{j=0}^{r-1} (-1)^j E_{\nu^j \alpha}. \quad (\text{C.21})$$

In the representation (C.18)

$$\begin{aligned} R_l &= \{ \tilde{\alpha} = \frac{1}{2}(e_j \pm e_k - (e_{2n+2-j} \pm e_{2n+2-k})) \}, \quad j = 1, \dots, n, \\ R_s &= \{ \tilde{\alpha} = \frac{1}{2}(e_j - e_{2n+1-j}) \}, \quad j = 1, \dots, n. \end{aligned} \quad (\text{C.22})$$

$$E_{\tilde{\alpha}} = \begin{cases} E_{jk} - E_{2n+2-k, 2n+2-j} & \tilde{\alpha} \in R_l, \\ E_{j, n+k} - E_{2n+2-k, 2n+2-j} & \tilde{\alpha} \in R_l, \\ E_{n+k, j} - E_{2n+2-j, 2n+2-k} & \tilde{\alpha} \in R_l, \\ \sqrt{2}(E_{j, n+1} - E_{n+1, 2n+2-j}) & \tilde{\alpha} \in R_s, \\ \sqrt{2}(E_{n+1, j} - E_{2n+2-j, n+1}) & \tilde{\alpha} \in R_s \end{cases} \quad (\text{C.23})$$

$$\mathfrak{t}_{\tilde{\alpha}}^1 = \begin{cases} E_{jk} + E_{2n+2-k, 2n+2-j} & \text{long roots,} \\ E_{j, n+k} + E_{2n+2-k, 2n+2-j} & \text{long roots,} \\ E_{n+k, j} + E_{2n+2-j, 2n+2-k} & \text{long roots,} \\ \sqrt{2}(E_{j, n+1} + E_{n+1, 2n+2-j}) & \text{short roots,} \\ \sqrt{2}(E_{n+1, j} + E_{2n+2-j, n+1}) & \text{short roots.} \end{cases} \quad (\text{C.24})$$

The twisted canonical basis in $\mathfrak{h} \subset \mathfrak{sl}(2n+1)$ is

$$\mathfrak{e}_k^0 = \frac{1}{\sqrt{2}}(e_k - e_{2n+1-k}), \quad \mathfrak{e}_k^1 = \frac{1}{\sqrt{2}}(e_k + e_{2n+1-k}), \quad (k = 1, \dots, n). \quad (\text{C.25})$$

The expressions (C.23)-(C.24) define the basis in $\mathfrak{g} = \mathfrak{sl}(2n+1)$ with the Killing form

$$\begin{aligned} (E_{\tilde{\alpha}}, E_{\tilde{\beta}}) &= 4\delta_{\tilde{\alpha}, -\tilde{\beta}}, & \tilde{\alpha} \in R_l, \\ (E_{\tilde{\alpha}}, E_{\tilde{\beta}}) &= 8\delta_{\tilde{\alpha}, -\tilde{\beta}}, & \tilde{\alpha} \in R_s, \\ (\mathbf{t}_{\tilde{\alpha}}^1, \mathbf{t}_{\tilde{\beta}}^1) &= 4\delta_{\tilde{\alpha}, -\tilde{\beta}}, & \text{long roots}, \\ (\mathbf{t}_{\tilde{\alpha}}^1, \mathbf{t}_{\tilde{\beta}}^1) &= 8\delta_{\tilde{\alpha}, -\tilde{\beta}}, & \text{short roots}, \\ (\mathbf{e}_k^m, \mathbf{e}_j^n) &= 4\delta^{m+n, r} \delta_{j, k}. \end{aligned} \quad (\text{C.26})$$

We pass from the canonical basis (C.15), (C.25) in \mathfrak{h}_0 to the basis of simple coroots

$$\tilde{\Pi}^\vee = \{H_{\tilde{\alpha}_j} \equiv H_j = \frac{2\tilde{\alpha}_j}{(\tilde{\alpha}_j, \tilde{\alpha}_j)}, \tilde{\alpha}_j \in \tilde{\Pi}, (C.4)\} \quad (\text{C.27})$$

Let Ξ be the system of fundamental weights of \mathfrak{g}

$$\Xi = \{\varpi_1, \dots, \varpi_l \mid \langle \varpi_j, \alpha_k \rangle = \delta_{jk}, \alpha_k \in \Pi\}. \quad (\text{C.28})$$

Then the fundamental co-weights $\tilde{\Xi}$ of the invariant subalgebra \mathfrak{g}_0 are defined as

$$\tilde{\varpi}_j^\vee = \begin{cases} \varpi_j^\vee, & \text{if } \nu(\alpha_j) = \alpha_j, \\ \sum_{m=0}^{r-1} \nu^m(\varpi_j^\vee) & \text{if } \nu(\alpha_j) \neq \alpha_j \end{cases} \quad (\text{C.29})$$

The system $\tilde{\Psi}^\vee = \{\tilde{\varpi}_j^\vee\}$ is dual to the system of simple roots $\tilde{\Pi}$ of \mathfrak{g}_0 ($\langle \tilde{\varpi}_j^\vee, \tilde{\alpha}_k \rangle = \delta_{jk}$).

Let \tilde{P}^\vee be the coweight lattice of \mathfrak{g}_0 generated by the fundamental coweights

$$\tilde{P}^\vee = \sum_{j=1}^n m_j \tilde{\varpi}_j^\vee, \quad m_j \in \mathbb{Z}, \quad (\text{C.30})$$

and $\tilde{Q}^\vee \subseteq \tilde{P}^\vee$ is the coroot lattice

$$\tilde{Q}^\vee = \sum_{j=1}^n m_j \tilde{\alpha}_j^\vee, \quad m_j \in \mathbb{Z}. \quad (\text{C.31})$$

It follows from (C.6) and (C.29) that both lattices are ν -invariant

$$\nu(\tilde{P}^\vee) = \tilde{P}^\vee, \quad \nu(\tilde{Q}^\vee) = \tilde{Q}^\vee. \quad (\text{C.32})$$

Moreover, $\tilde{P}^\vee \subset P$, while $\tilde{Q}^\vee \not\subset Q$.

Let

$$\tilde{C}^+ = \{x \in \mathfrak{h}_0^{\mathbb{R}} \mid \langle \tilde{\alpha}, x \rangle > 0\}. \quad (\text{C.33})$$

be the positive Weyl chamber and W_0 is the Weyl group generated by the reflections of the root system \tilde{R} . Then any regular element of $\Re \mathfrak{h}_0$ belongs to the W_0 -orbit of $\mathbf{u} \in \tilde{C}^+$.

By means of \tilde{Q}^\vee and W_0 define affine Weyl group \tilde{W}_a

$$\tilde{W}_a = W_0 \ltimes \tilde{Q}^\vee \quad (\text{C.34})$$

acting on $\Re \mathfrak{h}_0$ as

$$x \rightarrow x - \langle \tilde{\alpha}, x \rangle \tilde{\alpha}^\vee + k \tilde{\beta}^\vee, \quad \tilde{\alpha}^\vee, \tilde{\beta}^\vee \in \tilde{R}^\vee \quad k \in \mathbb{Z}. \quad (\text{C.35})$$

The Weyl alcoves are connected components of the set $\mathfrak{h}_0^{\mathbb{R}} \setminus \{\langle \tilde{\alpha}, x \rangle \in \mathbb{Z}\}$. Their closure are fundamental domains of the \tilde{W}_a -action. The group \tilde{W}_a acts transitively on the set of the Weyl alcoves.

By means \tilde{P}^{\vee} define the semidirect product

$$\tilde{W}'_a = W_0 \ltimes \tilde{P}^{\vee} \supseteq \tilde{W}_a. \quad (\text{C.36})$$

The connected components of the quotient $\Re \mathfrak{h}_0 / \tilde{W}'_a$ are alcoves

$$\tilde{C}'_{alc} = \Re \mathfrak{h}_0 / \tilde{W}'_a. \quad (\text{C.37})$$

The quotient group $\tilde{\mathcal{Z}} = \tilde{P}^{\vee} / \tilde{Q}^{\vee} \sim \tilde{W}'_a / \tilde{W}_a$. These groups have the structure

$$\tilde{\mathcal{Z}}(G) = \begin{cases} \mathbb{Z}_2 & \mathfrak{g} = A_{2n+1}, D_{n+1}, \\ Id & \mathfrak{g} = A_{2n}, D_4, E_6. \end{cases} \quad (\text{C.38})$$

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